



NAME		DATE	PERIOD
8-2 Practice			
Factoring	Using the Distributive	Property	
Factor each polynomial.			
1.64 - 40ab	2. $4d^2 + 16$	3. $6r^2s - 3$	irs ²
4. $15cd + 30c^2d^2$	5. $32a^2 + 24b^2$	6. 36xy ² –	48x ² y
7. $30x^3y + 35x^2y^2$	8. $9c^3d^2 - 6cd^3$	9. 75b ² c ³ +	$-60bc^3$
10. $8p^2q^2 - 24pq^3 + 16pq$	11. $5x^3y^2 + 10x^2y + 25x$	12. $9ax^3 + 18bx^2 + 24cx$	
$13.x^2 + 4x + 2x + 8$	14. $2a^2 + 3a + 6a + 9$	15. 4 <i>b</i> ² - 13	2b + 2b - 6
16. $6xy - 8x + 15y - 20$	17. $-6mn + 4m + 18n - 12$	18. 12a ² - 7	15ab - 16a + 20b
Solve each equation. Che	ck your solutions.		
19. $x(x - 32) = 0$	20. $4b(b + 4) = 0$	21. (y - 3)(y + 2) = 0
22. $(a + 6)(3a - 7) = 0$	23. $(2y + 5)(y - 4) = 0$	24. $(4y + 8)(3y - 4) = 0$	
25. $2z^2 + 20z = 0$	26. $8p^2 - 4p = 0$	27. $9x^2 = 27$	lx.
28. $18x^2 = 15x$. 29. $14x^2 = -21x$	30. $8x^2 = -$	26x

LANDSCAPING For Exercises 31 and 32, use the following information.

A landscaping company has been commissioned to design a triangular flower bed for a mall entrance. The final dimensions of the flower bed have not been determined, but the company knows that the height will be two feet less than the base. The area of the flower bed can be

represented by the equation $A = \frac{1}{2}b^2 - b$.

31. Write this equation in factored form.

32. Suppose the base of the flower bed is 16 feet. What will be its area?

33. PHYSICAL SCIENCE Mr. Alim's science class launched a toy rocket from ground level with an initial upward velocity of 60 feet per second. The height h of the rocket in feet above the ground after t seconds is modeled by the equation $h = 60t - 16t^2$. How long was the rocket in the air before it returned to the ground?

Practice	52		Glencoe Algebra 1	
	72II7	2		







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What is pseudocode?Starting Out With Visual Basic (8th Edition)What is a count-controlled loop?Starting Out with Python (3rd Edition)Write a program to print the value of EOF.C Programming LanguageRepeat Problem 3.84 for nitrogen gas.Fundamentals Of ThermodynamicsFind the change in length of side AB.Mechanics of Materials, 7th Edition Table of contents : Cover......Page 10 1.1: Proof by contradiction......Page 11 1.2: Algebraic fractions......Page 11 1.2: Algebraic fractions......Page 12 1.5: Algebraic division......Page 23 Mixed exercise: 1.....Page 28Chapter 2: Functions and graphs......Page 31 2.1: The modulus functions.....Page 32 2.2: Functions and mappings......Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| and y = f(|x|).....Page 45 2.5: y = |f(x)| 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423 iii Overarching themes 365 Inde can be applied alongside your learning and practice. 1. Mathematical argument, language and proof • Rigorous and consistent approach throughout • Notation boxes explain key mathematical language and symbols • Dedicated sections on mathematical sections on methods 2. Mathematical problem solving • Hundreds of problem-solving questions, fully integrated into the main exercises • Problem-solving cycle specify the problem interpret results collect information process and represent information • Dedicated modelling sections in relevant topics provide plenty of practice where you need it • Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model • Dedicated chapter in Statistics & Mechanics provide plenty of practice where you need it • Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model • Dedicated chapter in Statistics & Mechanics provide plenty of practice where you need it • Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model • Dedicated chapter in Statistics & Mechanics plenty of practice where you need it • Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model • Dedicated chapter in Statistics & Mechanics plenty of pl Year 1/AS explains the principles of modelling in mechanics Finding your way around the book. Each chapter starts with a list of objectives The Prior knowledge check helps make sure you are ready to start the chapter in The real world applications of the maths you are about to learn are highlighted at the start of the chapter with links to relevant questions in the chapter Overarching themes Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard Problem-solving boxes provide hints, tips and strategies, and Watch out boxes highlight areas where students often lose marks in their exams Exercises are packed with E Problem-solving questions to ensure you are ready for the exams Exam-style questions are flagged with P Each chapter ends with a Mixed exercise and a Summary of key points Challenge boxes give you a chance to tackle some more difficult questions Step-bystep worked examples focus on the key types of questions you'll need to tackle Each section begins with explanation and key learning points Every few chapters a Review exercise helps you consolidate your learning with lots of exam-style questions Two A level practice papers at the back of the book help you prepare for the real thing v Extra online content Extra online content Whenever you see an Online box, it means that there is extra online content available to support you. Differentiation 12A 1 a Examples of estimates of gradients: Gradient of tangent at x = -1 is $y_2 - y_1 3 - 1 = x_2 - x_1 (-1) - (-0.5) = -4$ Gradient of tangent at x = 0 is $y_2 - y_1 1 - (-1) = x_2 - x_1 (-0.5) - (0.5) = -2$ Gradient of tangent at x = 1 is $y_2 - y_1(-1) - (-1) = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 2 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1(-1) - 1 = x_2 - x_1 - 2 = 0$ Gradient of tangent at x = 3 is $y_2 - y_1$ Full worked solutions are available in SolutionBank. -10123-4-20242 c i Gradient of AD = y2 - y1 x2 - x1 0.8 - 0.19 = 0.6 - 0.8 = -1 iii Gradient of AB = y2 - y1 x2 - x1 0.8 - 0.51 = 0.6 - 0.7 = -0.859 (3 s.f.) d As the points move closer to A, the gradient tends to -0.75. 3 a i Gradient = 16 - 9 = 7 = 74 - 31 b The gradient of the curve at the point where x = p is 2p - 2. ii Gradient = 12.25 - 93.25 = 6.53.5 - 30.5 c Gradient of tangent at x = 1.5 is $y^2 - y_1(-1.7) - 0.3 = x^2 - x_10.5 - 2.5 = 12p - 2 = 2(1.5) - 2 = 1$ iii Gradient = 9.61 - 90.61 = 6.13.1 - 30.1 iv Gradient = 9.0601 - 90.0601 = 6.01 $3.01 - 3\ 0.01\ v$ Gradient = $(3 + h)^2 - 9\ (3 + h) - 3\ 2$ a Substituting x = 0.6 into = y 1 - x2 : y = 1 - 0.6\ 2 = 0.64 = 0.8, therefore the point A (0.6, 0.8) lies on the curve. 6h h + h2 = h h (6 + h) = h = 6 + h b Gradient of tangent at x = 0.6 is y2 - y1\ 1.1 - 0.8 = x2 - x1\ 0.2 - 0.6 = -0.75\ 12\ Differentiation, Mixed Exercise 1 2 dy x 2 = 6 \times 2 - 3 = -0.64 When 32 dx f(x + h) - f(x) 2h = 12 - 82210(x + h) - 10 x @ Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free. $3 = \lim h \rightarrow 0 = 11 h 2 4 2 2 10 x + 20 xh + 10h - 10 x 2 dy = \lim 6 \times 3 - 3 h h \rightarrow 0$, x 3 = = When $3 dx 2 20 xh + 10h 2 = \lim = 18 - h \rightarrow 0 h 27 h(20 x + 10h) 25 = \lim h a - 10 x 2 dy = 10 x + 20 xh + 10h - 10 x 2 dy = 10 x + 20 xh + 10h - 10 x 2 dy = 10 x + 20 xh + 10h 2 = 10 x + 20 xh + 10h 2 = 10 x + 20 xh + 10h 2 = 10 x + 20 xh + 10h - 10 x 2 dy = 10 x + 20 xh + 10h - 10 x 2 dy = 10 x + 20 xh + 10h 2 = 10 x + 20 xh +$ $-0 = 17 h 27 = \lim(20 x + 10h) 4$, $h \rightarrow 0$ The gradients at points A, B and C are 25 As $h \rightarrow 0$, $20x + 10h \rightarrow 20x 11 34$ and 17 27, respectively. So f'(x) = $20x 3y = 7x^2 - x 4^2 a$ A has coordinates (1, 4). dy 2 = 14x - 3x The y-coordinate of B is dx (1 + δx) 3 + $3(1 + \delta x) 3 + 3(1 + \delta x) 3 + 3($ $6\delta x \ 14x - 3x2 = 16$ Gradient of AB $3x^2 - 14x + 16 = 0$ y -y = 21(3x - 8)(x - 2) = 0 $x^2 - x^1 \ 8 \ 2 \ 3 \ x = or \ x = 2(\delta x) + 3(\delta x) + 6\delta x \ dy = 3x^2 - 11 \ \delta x \ dx \ 2 = (\delta x) + 3\delta x + 6 \ dy = 1$ when x d 2 b As $\delta x \rightarrow 0$, $(\delta x) + 3\delta x + 6 \ dy = 3x^2 - 11 \ \delta x \ dx \ 2 = (\delta x) + 3\delta x + 6 \ dy = 1$ when x d 2 b As $\delta x \rightarrow 0$, $(\delta x) + 3\delta x + 6 \ dy = 3x^2 - 11 \ \delta x \ dx \ 2 = (\delta x) + 3\delta x + 6 \ dy = 1$ when x d 2 b As $\delta x \rightarrow 0$, $(\delta x) + 3\delta x + 6 \ dy = 3x^2 - 11 \ \delta x \ dx \ 2 = (\delta x) + 3\delta x + 6 \ dy = 1$ point A is 6. $x^2 = 4x = \pm 231 - 22x = 2$, y = 2 - 11(2) + 1 = -13 When $2x + 3 + 3x = +3 + 3y = 3x^3 = 15x^2$ When x = -2, y = (-2) - 11(-2) + 12 The gradient is 1 at the points (2, -13) dy -3 = 6x - 2x = 6x - 3 and (-2, 15). $x dx f(x) = 10x^2 f'(x) = \lim h \rightarrow 0$ Download all the solutions as a PDF or quickly find the solution you need online When x = 1, $9 = x + 9x - 1 \times 9 = 2$ f'(x) = 1 - 9x = 1 - x + 22 dy = $6 \times 1 - 31$ dx = 4 © Pearson Education Ltd 2017. 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If you spot an error, please do contact us at so we can make sure it is corrected. viii 1 Algebraic methods Objectives After completing this chapter you should be able to: • Use proof by contradiction to prove true statements \rightarrow pages 2-5 \bigcirc Multiply and divide two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or subtract two or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or more algebraic fractions \rightarrow pages 5-7 \bigcirc Add or more denominator into partial fractions \rightarrow pages 12-13 \oplus Divide algebraic expressions \rightarrow pages 14-17 \oplus Convert an improper fraction form \rightarrow pages 12-13 \oplus Divide algebraic expressions \rightarrow pages 12-16 c 9x2 - 25 You can use proof by contradiction to prove that there is an infinite number of prime numbers. Very large prime numbers are used to encode chip and pin \rightarrow Example 4, page 3 transactions. \leftarrow Year 1, Section 1.3 2 Simplify fully the following algebraic fractions. $x^2 - 92$ $x^2 + 5x - 12a$ $bx^2 + 9x + 18 6x^2 - 7x - 32x - x - 30c$ \leftarrow Year 1, Section 7.1 -x 2 + 3x + 18 3 For any integers n and m, decide whether the following will always be odd, always be even, or could be either. a 8n b n-m c 3m d 2n - 5 - Year 1, Section 7.6 1 Chapter 1 1.1 Proof by contradiction is a disagreement between two statements, which means that both cannot be true. Proof by contradiction is a disagreement between two statements are contradiction are contradictin are contradiction are con you start by assuming it is not true. You then use logical steps to show that this assumption, or a contradiction of the assumption, or a contradiction of the assumption, or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement was true. Example Notation A statement that asserts the falsehood of another statement is called the negation of that statement. 1 Prove by contradiction that there is no greatest odd integer, n. n + 2 is also an integer and n + 2 > n n + 2 = odd + even = odd So there exists an odd integer greater than n. This contradicts the assumption that the greatest odd integer is n. You need to use logical steps to reach a contradicts your initial assumption. Therefore, there is no greatest odd integer. Finish your proof by concluding that the original statement must be true. Example 2 Prove by contradiction that if n2 is even, then n must be even. Assumption: there exists a number n such that n2 is even but n is odd so write n = 2k + 1 You can write any odd number in the form 2k + 1 where k is an integer. = (2k + 2 + 4k + 1)2(2k + 2k) + 1 So n2 is odd. This contradicts the assumption that n2 is even. n2 1)2 4k2 Therefore, if n2 is even then n must be even. All multiples of 2 are even numbers, so 1 more than a multiple of 2 is an odd number. Finish your proof by concluding that the original statement must be true. A rational number can be written as a , where a and b are integers. b \blacksquare An irrational number cannot be expressed in the form a , where a and b are integers. 2 b Notation \mathbb{Q} is the set of all rational number. Assumption: $\sqrt{2}$ is a rational number. a Then $\sqrt{2}$ = for some integers, a and b. b Also assume that this fraction cannot be reduced further: there are no common factors between a and b. a2 So 2 = 2 b 2 b This means that a2 must be even, so a is also even. If a is even, then it can be expressed in the form a = 2n, where n is an integer. So a2 =
2b2 becomes (2n)2 = 2b2 which means 4n2 = 2b2 or 2n2 = b2. This means that b2 must be even so b is also even. If a and b are both even, they will have a common factor of 2. This contradicts the statement that a and b have no common factors. _____ Therefore $\sqrt{2}$ is an irrational number. If a and b did have a common factor you could just cancel until this fraction was in its simplest form. Square both sides and make a2 the subject. We proved this result in Example 2. All even numbers are divisible by 2. Finish your proof by concluding that the original statement must be true. 4 Prove by contradiction that there are infinitely many prime numbers. Assumption: there is a finite number of prime numbers. List all the prime numbers that exist: p1, p2, p3, ..., pn You get a remainder of 1. So none of the prime numbers p1, p2, p3, ..., pn you get a remainder of 1. So none of the prime numbers p1, p2, p3, ..., pn is a factor of N. So N must either be prime or have a prime factor which is not in the list of all possible prime numbers. This is a contradiction. Therefore, there is an infinite numbers. This new number is one more than the product of the existing prime numbers. This contradicts the assumption that the list p1, p2, p3, ..., pn contains all the prime numbers. Conclude your proof by stating that the original statement must be true. 3 Chapter 1 Exercise P 1A 1 Select the statement must be true. 3 Chapter 1 Exercise P 1A 1 Select the statement must be true. even. P 2 Write down the negation of each statement. a All rich people are happy. b There are no prime numbers then (pq + 1) is a prime number. d All numbers of the form 2n - 1 are either prime numbers or multiples of 3. e At least one of the above four statements is true. P 3 Statement: If n2 is odd then n is odd, a Write down the negation of this statement, b Prove the original statement by contradiction. P 4 Prove the following statements by contradiction. P 4 Prove the following statement by contradiction. E/P 5 a Prove that if ab is an irrational number. (3 marks) b Prove that if a + b is an irrational number. (3 marks) b Prove that if a + b is an irrational number. means of a counterexample that this statement is not true. P E/P 6 Use proof by contradiction to show that there exist no integers a and b for which 21a + 14b = 1. Hint 7 a Prove by contradiction that $\sqrt{3}$ is an irrational number. (3 marks) 4 (1 mark) Assume the opposite is true, and then divide both sides by the highest common factor of 21 and 3k + 2. Algebraic methods P E/P 8 Use proof by contradiction to prove the statement: 'There are no integer solutions to the equation $x^2 - y^2 = 2'$ ' 3 Hint You can assume that x and y are positive, since $(-x)^2 = x^2$. 9 Prove by contradiction that $\sqrt{2}$ is irrational. (5 marks) 10 This student has attempted to use proof by contradiction to show that there is no least positive rational number. Let this least positive rational number. Let this least positive rational number be n. a As n is rational, n = 0 where a and b are integers. b a a-b n -1= _ 1 = _ b b a - b Since a and b are integers, _____ is a rational number that is less than n. b This contradicts the statement that n is the least positive rational number. Therefore, there is no least positive rational number. Problem-solving You might have to analyse student working like this in your exam. The question savs, 'the error', so there should only be one error in the proof. a Identify the error in the student's proof. (1 mark) b Prove by contradictions that there is no least positive rational number. (5 marks) 1.2 Algebraic fractions work in the same way as numeric fractions. You can simplify them by cancelling common factors and finding common denominators. To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators. Example 5 Simplify the following products: $35 a c a \times b \times 59 b a 1 3 51 1 \times 11 a \times c = 1 = 931 \times 33 15 1a c 1 \times c c b \times c = 2 (x + 1)(x - 1) 1 3 = 2 (x - 1) x + 1 3 c \times c = 2 (x - 1) x + 1 3 (x - 1) x + 1 x$ 1) $x^2 - 1 + 3 + 1 = x$ Chapter 1 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction. Example 6 Simplify: a a a _ \div _ b c x + 2 3x + 6 b _ \div _ = _ x _ b c b a1 Multiply the first fraction by the reciprocal of the second fraction. Cancel the common factor a. 1×c = _ b×1 c _ = b x + 2 3x + 6 b _ \div $\begin{array}{c} \text{chapter 1} \\ \hline \text{b} \text{chapter 1} \\ \hline \text{b} \text{chapter 1} \\ \hline \text{cha$ marks) Hint Simplify and then solve the logarithmic equation. \leftarrow Year 1, Section 14.6 b Given that ln ((x 2 + 2x - 24)(x 2 - 3x)) = 2 + ln ((2x 2 + 10x)(x 2 + 3x - 18)) find x in terms of e. E/P 2x 2 - 3x - 2 x - 2 6 f(x) = _____ \div ____ 2 for a diagonal control of the logarithmic equation. \leftarrow Year 1, Section 12.5 (4 marks) Hint 2x 2 + 13x + 6 a Show that f(x) = _____ 2 Differentiate each term separately. \leftarrow Year 1, Section 12.5 (4 marks) b Hence differentiate f(x) and find f9(4). (3 marks) \blacksquare To add or subtract two fractions, find a common denominator. Example 7 Simplify the following: a b 1 3 b __ + __ a __ + __ 3 4 2x 3x 1 __ a 4 x __ 4 3 3 + __ 4 4 9 = __ + __ 12 12 2 1 c ___ - ___ x+3 x+1 3 4 d ___ - ___ x+1 x2 - 1 3 x __ 3 13 = ____ x = __ 12 + $12 \text{ b a b}_{-+} 2x 3x 3a 2b = -+ 6x 6x 3a + 2b = 6x 21 \text{ c}_{--} (x + 3) (x + 1) 2(x + 1) 1(x + 3) = - (x + 3)(x + 1) (x + 3) (x + 1) (x + 3) = - (x + 3)(x + 1) (x + 3) = - (x + 3)(x + 1) \text{ the lowest common multiple of } 2x and 3x is 6x. Multiply the first fraction by _33 and the second fraction by _22 The lowest common multiple is (x + 3)(x + 1), so that the denominators are (x + 3)(x + 1) - 1(x + 3) = - (x + 3)(x + 1) \text{ Subtract the numerators. } 2x + 2 - 1x - 3 = - (x + 3)(x + 1) \text{ Subtract the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x +
3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ Simplify the numerators. } 7 \text{ Chapter 1 4x 3 d} = - x + 3 \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the lowest common multiple is } (x + 3)(x + 1) \text{ the$ x + 1 $x^{2} - 1 4x^{3} = - x^{2} + 1 (x + 1)(x - 1)$ x + 1 (x + 1)(x - 1) x + 1 (x + 1)(x -(x + 1)(x - 1) Exercise ____a _____x 2 + 2x + 1 $\overline{2x - 3x} + 2x - 155$ Express each of the following as a fraction in its simplest form. $3\ 2\ 4\ 2\ 1\ 1\ a$ + $_$ + $_$ b $_$ + $_$ + $_$ + $_$ b $_$ + $_$ + $_$ b $_$ + $_$ + $_$ b $_$ + $_$ + $_$ + $_$ + $_$ + $_$ b $_$ + denominator can be split into two separate fractions with linear denominators. This is called splitting it into partial fractions. 5 $(x + 1)(x - 4) \equiv A$ x+1 + B x-4 The denominator contains two linear factors: (x + 1) and (x - 4). A and B are constants to be found. The expression is rewritten as the sum of two partial fractions. Links ≡ + (x – identical to + x-3x+1(x-3)(x+1) Add the two fractions. To find A substitute x = 3. This value of x fractions can also be used when there are more than two distinct linear factors in the denominator. 7 For example, the expression (x - 2)(x + 6)(x + 3) C A B can be split into + + x-2x+6x+3 The constants A, B and C can again be found by either substitution or by equating coefficients. Example Watch out This $\frac{1}{1} + \frac{1}{1} + \frac{1$ $8x + 2 \text{ E D F 5 Given that, for } x < -1, _ = + _ + _ + _ , \text{ where D, E and F are } x(2x + 1)(3x - 2) \times 2x + 1 3x - 2 \text{ constants. Find the values of D, E and F. (4 marks) 6 Express the following as partial fractions: <math display="block">2x^2 - 12x - 26 \text{ a} _ (x + 1)(x - 2)(x + 5) \text{ P} - 10x^2 - 8x + 2 \text{ b} _ x(2x + 1)(3x - 2) - 5x^2 - 19x - 32x^2 - 15x - 8 \text{ Express the following as partial fractions: } 6x^2 + 7x - 3 \text{ a} _ x^3 - x 8x + 9 \text{ b} _ 10x^2 + 3x - 4 \text{ Hint First factorise the denominator. Challenge } 5x^2 - 15x - 8 \text{ Express } _ as a sum of fractions with linear denominators. } x^3 - 4x^2 + x + 611 \text{ Chapter 1 1.4 Repeated factors } A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions. In this case, there is a special method for dealing with the repeated linear factor. <math display="block">2x + 9 _ (x - 5)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ The denominator } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ The denominator } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ The denominator } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ The denominator } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ The denominator } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ Chapter 1 and } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ Chapter 1 and } x + 10(x - 2)(x + 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ 2x - 5x + 3(x + 3) \text{ Chapter 1 and } x + 10(x - 2)(x - 3)^2 \text{ A and B and C are constants to be found. } C \text{ A B = } + _ + _ + _ + _ + = 2x - 5x + 3(x + 3) \text{ Chapter 1 and } x + 10(x - 2)(x$ three linear factors: (x - 5), (x + 3) and (x + 3). (x + 3) is a repeated linear factor. The expression is rewritten as the sum of three partial fractions. Notice that (x - 5), (x + 3) and $(x + 3)^2$ are the denominators. Example 10 C 11x 2 + 14x + 5 A B can be written in the form _____ + ____2 + _____, where A, B and C Show that ______ 2x + 1 (x + 1) 2(2x + 1) are constants to be found. Let 11x2 + 14x + 5+ 14x + 5 = A (x + 1)(2x + 1) + B(2x + 1) + B(2x + 1) + C(x + 1)2 = (x + 1)(2x + 1)(x + 1)2(2x + 1) (x + 1)2(2x + 1) + B(2x + 1) + B(2x + 1) + C(x + 1)2 = (x + 1)(2x + 1) + B(2x + 1) +(x + 1)2(2x + 1) Add the three fractions. Hence 11x2 (x + = - (x + 1) (x + 1)2 (2x + 1) So A = 4, B = -2 and C = 3. 12 To find C substitute x = - 12 Equate terms in x2 in (1). Terms in x2 are A × 2x2 + C × x2. Substitute C = 3. Finish the question by listing the coefficients. Online Check your answer using the simultaneous equations function on your calculator. 1)2(2x + 1) 4 3 2 +Algebraic methods Exercise E 1E 3x 2 + x + 1 1 f(x) = _____, x \neq 0, x \neq -1 x 2(x + 1) C A B Given that f(x) can be expressed in the form + 2 + - 1, find the values of x x + 1 A, B and C. E -x 2 - 10x - 5, x $\neq -1$, x $\neq 1 2$ g(x) = ______ (x + 1) 2(x - 1) D F E Find the values of the constants D, E and F such that g(x) = ______ + ____ 2 + _____ = 2 + _____ = 2 + _____ = 2 + ______ () x+1x+1x-1 E E E P (4 marks) Q 2x 2 + 2x - 18 P R = + + + 2, where P, Q and R are constants, 3 Given that, for x < 0, 2x 2 + 2x - 1 E can be written in the form + 2 + where C, D and E 4 Show that 3 2 x x - 1 x - x are constants, 3 Given that, for x < 0, 2 x 2 + 2x - 1 E can be written in the form + 2 + where C, D and E 4 Show that 3 2 x x - 1 x - x are constants are constants are constants, 3 Given that, for x < 0, 2 x 2 + 2x - 1 E can be written in the form + 2 + where C, D and E 4 Show that 3 2 x x - 1 x - x are constants are constants are constants are constants. A more constants A and B such that p(x) = + 2 x + 2 (x + 2) E (4 marks) C 10x 2 - 10x + 17 A B 6 2 = + + + 2, x > 3 x - 3 2x + 1 (2x + 1)(x - 3) (x - 3) Find the values of the constants A, B and C. (4 marks) C 39x 2 + 2x + 2 (x + 2) E (4 marks) C 39x 2 + 2x + 2 (x + 2) E (4 marks) C 39x 2 + 2x + 2 (x + 2) E (4 marks) C 39x 2 + 2x + 2 (x + 2) E (4 marks) C 39x 2 + 2x + 2 (x + 2) E (4 marks) C 39x 2 + 2x + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x + 2) E (4 marks) C 39x 2 + 2 (x +
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2 where x + 5 3x - 1 (3x - 1) (x + 5)(3x - 1) A, B and C are constants to be found. (4 marks) 8 Express the following as partial fractions: 4x + 1 a2 x + 10x + 25 6x2 - x + 2 b 4x3 - 4x2 + x 13 Chapter 1 1.5 Algebraic division \blacksquare An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions. x + 5x + 8x + 5x - 9and are both improper fractions. $2 \ 3 \ x-2 \ x3 \ -4x2 \ +7x \ -3$ The degrees of the numerator and denominator are equal. The degree of the numerator is greater than the degree of the denominator. Notation \blacksquare You can either use: • algebraic division • or the relationship $F(x) = Q(x) \times divisor + remainder$ The degree of a polynomial is the largest exponent in the expression. For example, x 3 + 5x - 9 has degree 3. Watch out The divisor and the remainder can be numbers or functions of x. to convert an improper fraction into a mixed fraction. Method 1 Use algebraic long division to show that: F(x) x + 5x + 8 $2x-2 \equiv x+7+Q(x)$ 22 remainder x-2 divisor Method 2 Multiply by (x-2) and compare coefficients to show that: $F(x)Q(x) \ge 2 + 5x + 8 \equiv (x+7)(x-2) + 22$ remainder divisor Example 11 $x_3 + x_2 - 7$ D Given that = Ax 2 + Bx + C + x_{-3} , find the values of A, B, C and D. $x_{-3}x_{-3}$ Using algebraic long division: $x^2 + 4x x - 3x^3 + x^2 x^3 - 3x^2 4x^2 4x^2 14 + 12 + 0x - 7 + 0x - 12x 12x - 7 12x - 36 29$ Problem-solving Solving this problem using algebraic long division will give you an answer in the form asked x-329 = x2 + 4x + 12 + x-3 So A = 1, B = 4, C = 12 and D = 29. The divisor is (x - 3) so you need to write the remainder as a fraction with denominator (x - 3). It's always a good idea to list the value of each $= x^{2} + 4x + 12x - 3$ with a remainder of 29 $x^{3} + x^{2} - 7$ for in the question. Algebraic methods $x_3 + x_2 - 7$ So unknown asked for in the question. Example 12 Given that $x 3 + x 2 - 7 \equiv (Ax 2 + Bx + C)(x - 3) + D$, find the values of A, B, C and D. Let $x = 3: 27 + 9 - 7 = (9A + 3B + C) \times (0 - 3) + D - 7 = -3C + D - 7 =$ coefficients in x3: 1=A Compare coefficients in x2: 1 = -3A + B = 1 = -3 + B Therefore A = 1, B = 4, C = 12 and D = 29 and we can write x 3 + x 2 - 7 = (x 2 + 4x + 12)(x - 3) + 29 This can also be written as: x3 + x2 - 7 x-3 Problem-solving The identity is given in the form $F(x) \equiv Q(x) \times \text{divisor} + \text{remainder so solve the problem by}$ equating coefficients. Set x = 3 to find the value of D. Set x = 0 and use your value of D to find the remaining values by equating coefficients of x^3 and x^2 . Remember there are two x^2 terms when you expand the brackets on the RHS: x^3 terms: LHS = x^3 , RHS = $Ax^3 x^2$ terms: LHS = x^2 , RHS = $(-3A + B)x^2 29 \equiv x^2 + 4x + 4x + 10x^2 +$ x-3 Example 13 x 4 + x 3 + x - 10 f(x) = _____ and find the values of A, B, C, D and E. 2x + 2x - 3 15 Chapter 1 Using algebraic long division: $x^2x^2 - x + 2x - 3x^4 + x^3 + x^4 + 2x^3 - x^3 + x^4 + +$ $x^{2} + 2x - 3Dx + E$ Show that f(x) can be written as Ax 2 + Bx + C + x + 2x - 3 Watch out When you are dividing by a quadratic expression, the remainder can be a constant or a linear expression. The degree of (-12x + 5) is smaller than the degree of $(x^2 + 2x - 3)$ so stop your division here. The $0x^{2} + x - 10\ 3x^{2}\ 3x^{2} + x\ 2x^{2} + 3x\ 5x^{2} - 2x - 10\ 5x^{2} + 10x - 15\ -12x + 5\ -12x + 5 \equiv x\ 2 - x + 5 + 5x^{2} + 5x^{$ remainder is -12x + 5. Write the remainder as a fraction over the whole divisor. So A = 1, B = -1, C = 5, D = -12 and E = 5. 1F Exercise E 1 E 2 E 3 x + 2x + 3x - 4 $3 2 D \equiv Ax 2 + Bx + C + x+1 x+1$ Find the values of the constants A, B, C and D. d 2x 3 + 3x 2 - 4x + 5 Given that find the $\equiv ax 2 + bx + c +$ x-2 Show that f(x) can be written in the form px 2 + qx + r and find the values of p, q and r. E 4 nx + p 2x 2 + 4x + $5 \equiv m + 2$ find the values of m, n and p. Given that $2 \times -1 \times 2 - 1 \times 5$ Find the values of the constants A, B, C and D in the following identity: $8 \times 3 + 2 \times 2 + 5 \equiv (A \times + B)$ values of a, b, c and d. x+3x+3x3 - 8f(x) = $3 2 x^2 + 2x - 1$ (4 marks) (4 marks) (4 marks) (4 marks) (4 marks) Cx + D = Ax + B + x + 2x - 1 Find the values of the constants A, B, C and D. (4 marks) sx + tx + 3x - 4. Show that g(x) can be written in the form px + 2 + qx + r + 1(2x 2 + 2) + Cx + D E E E 6 7 8 16 4x - 5x + 3x - 14q(x) = $x^{2} + 1$ and find the values of p, q, r, s and t. (4 marks) $2x^{4} + 3x^{3} - 2x^{2} + 4x - 6 dx + e \equiv ax^{2} + bx + c + c$ find the values Given that $x^2 + x - 2x^2 + x - 2$ of a, b, c, d and e. (5 marks) Algebraic methods E 9 Find the values of the constants A, B, C, D and E in the following identity: $3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (Ax^2 + x^2 - 2x^2)^2$ Bx + C $(x^2 - 3) + Dx + E E/P 10$ a Fully factorise the expression $x^4 - 1$. -1 b Hence, or otherwise, write the algebraic fraction in the form x+1 (ax + b)(cx2 + dx + e) and find the values of a, b, c, d and e. x4 (5 marks) (2 marks) (4 marks) In order to express an improper algebraic fraction in partial fractions, it is first necessary to divide the \equiv A + _____ + ____, find the values of A, B and C. x-1 x-2 (x - 1)(x - 2) Multiply out the denominator on the LHS. 3x2 · numerator by the denominator. Remember an improper algebraic fraction is one where the degree of the numerator is greater than or equal to the degree of the denominator. Example 14 C 3x 2 - 3x - 2 B Given that $(x - 1)(x - 2)x^2 - 3x + 23x^2 - 3x - 2$ $3 \equiv x^2 - 3x + 23x^2 - 3x - 23x^2 - 9x + 66x - 8$ Divide the denominator into the numerator. It goes in 3 times, with a remainder of 6x - 8. Therefore $6x - 83x^2 - 3x - 2$ $3x - 2 \equiv$ $\equiv 3 + (x - 1)(x - 2) x^2 - 3x + 2 6x - 8 \equiv 3 +$ $(x - 1)(x - 2) 6x - 8 \equiv B(x - 2) + C(x - 1)$ Let x = 2: $12 - 8 = B \times 0 + C \times 1$ C=4 Let x = 1: $6 - 8 = B \times -1 + C \times 0$ B=2 - 3x - 2 $(x - 1)(x - 2) (x - 1) (x - 2) B(x - 2) + C(x - 1) \equiv$ $3x2 (x - 1)(x - 2) \equiv 3 + \equiv 3 + 6x - 8$ (x - 1)(x - 2) 2as a mixed fraction. (x - 1)(x - 2) Factorise $x^2 - 3x + 2$. The denominators must be (x - 1) and (x - 2). Add the two fractions. The numerators are equal. Substitute x = 2 to find C. Substitute x = 1 to find B. Write out the full solution. (x - 2) Finish the question by listing the coefficients. 17 $3x^2 - 3x - 2$ Write and C = 4 + 4Chapter 1 1G Exercise E E E 1 2 3 C x 2 + 3x - 2 B g(x) = . Show that g(x) can we written in the form A + + x-1x-2(x + 1)(x - 3) and find the values of the constants A, B and C. (4 marks) C x 2 - 10 B Given that , find the values of the constants A, B x-2x+1(x-2)(x+1) and C. (4 marks) A, B, C and D in the following identity: x - x - x - 3 4x - 1 E 4 E 5 E P 6 7 C D = Ax + B + + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + Bx + x - 1 C -3x3 - 4x2 + 19x + 8 D can be expressed in the form A + x - 1 C -3x3 - 4x2 + 19x + 18 D can be expresse Find the values of the constants A, B, C and D in the following identity: x - x - x - 3 $x3 - 4x2 + 4x C 6x 3 - 7x 2 + 3 D \equiv Ax + B + ____ + ____$ x2 + 3x - 4 E 8 E 9 x4 - 4x3 + 9x2 - 17x + 12 b , find the values of the constants Given that 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (6 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (7 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (7 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (8 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (8 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (8 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (8 marks) 8x + 1q(x) = 3x - 5x + 23x + 11x - 10 A, B, C and D. (8 marks) 8x + 1q(x) = 3x - 5x +
11x - 10express the following as partial fractions: $4x^2 + 17x - 11a$ $x^2 + x - 2$ (6 marks) D E Show that h(x) can be written as Ax 2 + Bx + C + _____ + ____ and find the values of x+2 x-1 A, B, C, D and E. (5 + 2 and find the 2x - 1 (2x - 1) values of the constants A, B, C and D. E (5 marks) x 4 + 2x 2 - 3x + 8 10 h(x) = 4x + 1 C D Show that q(x) can we written in the form Ax + B + Bmarks) 18 Algebraic methods Mixed exercise 1 E/P 1 Prove by contradiction that $\sqrt{2}$ is an irrational number. P 2 Prove that if g2 is an irrational number. A simplify: 1x - 42x + 8a × $6x^2 - 16E/P 4x^2 - 3x - 10$ $6x^2 + 24b$ $\times 3x 2 - 21 x 2 + 6x + 8$ (5 marks) 4x 2 + 12x + 9 $\div x 2 + 6x 2x 2 + 9x - 18 4x 2 - 8x x 2 + 6x + 5 \times$ a Simplify fully $2x - 3x - 4 2x 2 + 10x (3 \text{ marks}) \text{ b Given that } \ln ((4x 2 - 8x)(x 2 + 6x + 5)) = 6 + \ln ((x 2 - 3x - 4)(2x 2 + 10x)) \text{ find x in terms of e. } (4 \text{ marks}) \text{ E/P } 5 x 2 - 3x 4x 3 - 9x 2 - 9x g(x) = 6 + \ln ((x - 3x - 4)(2x - 3x - 4)(2x$ 13x - 5 a Show that g(x) can be written in the form as 2 + bx + c, where a, b and c are constants to be found. (4 marks) b Hence differentiate g(x) and find g9(-2). (3 marks) (4 marks) E = 5x + 3 + 2 + 3x - 10 + $x - 1 x^{2}$ $+ 2x - 3x^{2} + 3x + 3$ Show that f(x) =Find the values of the constants P, Q and R. E 16x – 1 D E 10 Show that can be put in the form $Ax^2 + Bx + C + 2x + 1$ Find the values of the constants A, B, C and D. (5 marks) $x^4 + 2D \equiv Ax^2 + Bx + C + 2x + 1$ where A, B, C and D are constants to 15 Show that 22 x - 1 x - 1 be found. (5 marks) $x^4 D E 16$ Show that $\equiv Ax 2 + Bx + C + +$ 2 2 x - 1 (x - 1) x - 2x +1 (5 marks) Find the values of the constants A, B, C, D and E. E 2x 2 + 2x - 3x + 2x - 317 h(x) = 2 C B Show that h(x) can be written in the form A + + that f(x) = 2x3 + 9x2 + 10x + 3: (5 marks) (5 marks) (5 marks) a show that -3 is a root of f(x) 10 b express _____ as partial fractions. f(x) Challenge Hint The line L meets the circle C with centre O at exactly one point, A. Prove by contradiction that the line L is perpendicular to the radius OA. 20 In a right-angled triangle, the side opposite the right-angle is always the longest side. A L O C Algebraic methods Summary of key points 1 To prove a statement by contradiction of the assumption leads to something impossible (either a contradiction of the assumption or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement was true. a 2 A rational number can be written as __, where a and b are integers. b 3 To multiply fractions, cancel any common factors, then multiply the denominators. 4 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction. 5 To add or subtract two fractions; find a common denominator can be split into two separate fractions with linear denominators. This is called splitting it into partial fractions; 5 A B = + (x + 1)(x - 4) x + 1 x - 4 7 The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator: C B A 7 = + + (x - 2)(x + 6)(x + 3) x - 2 x + 6 x + 3 8 A single fraction with a repeated linear factor in the denominator can be split into two or more separate = + + (x - 5)(x + 3) 2 x - 5 x + 3 (x + 3) 2 9 An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions. 10 You can either use: • algebraic division • or the relationship $F(x) = Q(x) \times divisor + remainder to convert an improper fraction into a mixed fraction. 21 2 Functions and graphs Objectives After completing this chapter you should be able to: • Understand and use the modulus function <math>\rightarrow$ pages 23-27 • Understand mappings and functions, and use domain \rightarrow pages 27-32 and range \bullet Combine two or more functions to make a composite \rightarrow pages 32-35 function \bullet Know how to find the inverse of a function graphically \bullet Sketch the graphs of the modulus functions $y = |f(x)| \rightarrow$ pages 40-44 and $y = f(|x|) \bullet$ Apply a combination of two (or more) transformations to \rightarrow pages 44-48 the same curve \bullet Transform the modulus function \rightarrow pages 48-52 Prior knowledge check 1 Make y the subject of each of the following: 2y + 8x a 5x = 9 - 7y b p =______ 5 c $5x - 8y = 4 + 9xy \leftarrow$ GCSE Mathematics 2 Write each expression in its simplest form. 1 a (5x - 3) 2 - 4b ______ 2(3x - 5) - 4x + 4 _____ $+ 5x + 2 \leftarrow$ GCSE Mathematics c ______ x + 4 _____ - 3x+2 3 Sketch each of the following graphs. Label any points where the graph cuts the x- or y-axis. a y = b y = x(x + 4)(x - 5) ex c y = sin x, 0 < x < 360° 4 f(x) = x 2 - 3x. Find the values of: a f(7) 22 \leftarrow Year 1 b f(3) c f(-3) \leftarrow Year 1 b f(3) c f(-3) d f(enemy encoded a message they used a function. The modulus function that would decode the message. Functions and graphs 2.1 The modulus function that would decode the message. Function is also |-5| = 5. Notation The modulus function is also known as the absolute value function. On a calculator, the button is often labelled 'Abs'. A modulus function of the type $y = |f(x)| \cdot When f(x) > 0$, $|f(x)| = -f(x) \cdot When f(x) > 0$, $|f(x)| = -f(x) \cdot When f(x) > 0$, $|f(x)| = -f(x) \cdot When f(x) > 0$, $|f(x)| = -f(x) \cdot When f(x) = -1$ -5| = |15 - 15| = |-15| = 151 4 Example 5 12 7 7 6.5 is a positive number. Work out the value inside the modulus. 2f(x) = |2x - 3| + 1 = |7| + 1 = 7 + 1 = 8 Watch out The modulus function acts like a pair of brackets. Work out the value inside the modulus function acts like a pair of brackets. function first. b f(-2) = |2(-2) - 3| + 1 = |-7| + 1 = 7 + 1 = 8 c $f(1) = |2 \times 1 - 3| + 1 = |-1| + 1 = 2$ To sketch the graph of y = |ax + b|, sketch Online Use your calculator to work out values of modulus functions. y = x + b then reflect the section of the graph below the x-axis in the x-axis. O $y = |x| \times y$ reflected in the x-axis O x = 23Chapter 2 Example 3 y Online Sketch the graph of y = |3x - 2|. y = |3x - 2| 2 O Example 2 3 x Step 2 For the part of the line below the x-axis (the negative values of y), reflect in the x-axis. For example, this will change the y-value -2 into the y-value 2. You could check your answer using a table of values: x - 1 0 1 2 y = |3x - 2| 5 2 1 4 4 Solve the equation. A is the point of y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. By A y=5 y = |2x - 1| = 5. intersection on the original part of the graph. At A, 2x - 1 = 5 2x = 6 x = 3 At B, -(2x - 1) = 5 -2x + 1 = 5 2x = -4 x = -2 The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3 and x = -2. The solutions are x = 3. algebraically by considering the positive argument and the negative argument separately. Functions and graphs y Example 5 Online 1 Solve the equation |3x - 5| = 2 - x. 2 x y = 2 - 21 x The sketch shows there are two solutions, at A and B, the points of intersection. A 1 At A: 3x - 5 = 2 - x 27 x = 7 2 x = 2 1 At B: -(3x - 5) = 2 - x 27 x = -3 2 6 x = 5 6 The solution on the original part of the graph. 1 When f(x) < 0, |f(x)| = -f(x), so - 5 Example x Explore intersections of straight lines and modulus graphs using technology. This is the solution on the original part of the graph. 1 When f(x) < 0, |f(x)| = -f(x), so - 5 Example x Explore intersections of straight lines and modulus graphs using technology. This is the solution on the original part of the graph. 1 When f(x) < 0, |f(x)| = -f(x), so - $(3x - 5) = 2 - x^2$ gives you the second solution. This is the solution on the reflected part of the graph. 6 Solve the inequality |5x - 1| = 3x to find the x-1 = $3x + 1 = 3x^2 + 1 =$ coordinates of the points of intersection, A and B. This is the intersection on the original part of the graph. Consider the negative argument to find the point of intersection are $x = __{and 21 x} =$ values of 3 b |-0.28| a 4 || b f(10) || 5 3 d - 7 8 e $|20 - 6 \times 4|$ c f(-6) b g(-5) c g(8) Sketch the graph of each of the following. In each case, write down the coordinates of any points at which the graph meets the coordinates of any points at which the graph meets the coordinate axes. 1 a y = |x - 1| b y = |2x + 3| c y = |4x - 7| d y = x - 52 e y = |7 - x| f y = |6 - 4x| Hint y = -|x| is a reflection of y = |x| in the x-axis. \leftarrow Year 1, Chapter 4 g y = -|x| h y = -|3x - 1|
5 | 7 E 8 9 26 | 3 g(x) = 4 - __x and h(x) = 5 2 a On the same axes, sketch the graphs of y = g(x) and y = h(x). 3 b Hence solve the equation $4 - _x = 5$. 2 Solve: x-5 b ___ = 1 c |4x + 3| = -2 a |3x - 1| = 5 2 4 - 5x x d |7x - 3| = 4 e ___ = 2 f __ - 1 = 3 3 6 | 6 f | 4 2 × 2 - 3 × 7| g(x) = |x2 - 8x|. Write down the values of: a g(4) 4 c |3 - 11| f(x) = |7 - 5x| + 3. Write down the values of x satisfy the inequality. y = |5x - 1| is above 1 1 y = 3x when $x > _$ or $x < _$. You could write the 2 8 1 1 solution in set notation as $\{x : x > _\} \cup \{x : x < _\}$. 2 8 2A Exercise 1 Problem-10 A student attempts to solve the equation |3x + 4| = x. The student writes the following working: 3x + 4 = x + 4 = -2x = -2 and x = -2 and -|3x + 4| < 2x - 9. E E/P (4 marks) 12 Solve the inequality |2x + 9| < 14 - x. 1 13 The equation. Problem-solving The solution to the equation. Problem-solving The solution must be at the value of k. (2 marks) b State the solution to the equation. Problem-solving The solution must be at the value of k. (2 marks) b State the solution to the equation. Problem-solving The solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (2 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the solution must be at the value of k. (3 marks) b State the value of k. (3 marks) b S 1 - x a On the same axes, sketch graphs of y = f(x) and y = g(x). b Use your sketch to find all the solutions to |x 2 + 9x + 8| = 1 - x. 2.2 Functions and mappings A mapping transforms one set of numbers. The mapping can be described in words or through an algebraic equation. It can also be represented by a graph. A mapping is a function if every input has a distinct output. Functions can either be one-to-one function A B not a function _ Many mappings can be made into functions by changing the domain. Consider $y = \sqrt{x}$: y Notation The domain is the set of all possible inputs for a mapping. y = x O x The range is the set of all possible outputs for the mapping. 27 Chapter 2 If the domain were all of the real numbers, \mathbb{R} , then $y = \sqrt{x}$ would not be a function because values of x less than 0 would not be mapped anywhere. If you restrict the domain to x > 0, every element in the domain is mapped to exactly one element in the range. Notation You can also write this function as: We can write this function together with its $f: x \mapsto \sqrt{x}$, $x \in \mathbb{R}$, x > 0, \sqrt{domain} as f(x) = x, $x \in \mathbb{R}$, x > 0, \sqrt{domain} as f(x) = x, $y \neq 2x + 5$ d 1 y = x y = 2x + 5 d 1 y = x + 5= x2 - 1 y y O O x x - x O x x a i Every element in set A gets mapped to one value of y, so the mapping is one-to-one. ii The mapping is one-to-one. ii The mapping is one-to-one. ii x = 0 does not get mapped to a value of y so the mapping is not a function. You couldn't write down a single value for f(9). For a mapping to be a function, every input in the domain must map onto exactly one output. The mapping in part c could be a function, every input in the domain must map onto exactly one output. The mapping in part c could be a function if x = 0 were omitted from the domain. You could 1 write this as a function as f(x) = -x, $x \in \mathbb{R}$, $x \neq 0$. d i On the graph, you can see that x and -x both get mapped to the same value of y. Therefore, this is a many-to-one mapping is a function. Example 8 Find the range of each of the following functions: a f(x) = 3x - 2, domain $\{x \in \mathbb{R}, 0 < x < 3\}$ State if the functions are one-to-one or many-to-one or many-to-one mapping. one. 28 x Functions and graphs a f(x) = 3x - 2, $\{x = 1, 2, 3, 4\}$ 1 1 2 4 3 7 4 10 The domain contains a finite number of elements, so you can draw a mapping diagram showing the whole function. Range of f(x) is $\{1, 4, 7, 10\}$. f(x) is one-to-one. b $g(x) = x^2$, $\{-5 < x < 5\}$ y 25 -5 y = g(x) 5 O The domain is the set of all the x-values that correspond to points on the graph. The range is the set of y-values that correspond to points on the graph. x Range of g(x) is 0 < g(x) < 25. g(x) is many-to-one. 1 c h(x) = x {x $\in \mathbb{R}$, 0 < x < 3} y = h(x) 1 Calculate h(3) = x to find the minimum value in 3 1 the range of h. As x approaches 0, x < 3, so there is no maximum value in the range of h. 1 3 O 3×1 Range of h(x) is h(x) > ____ 3 h(x) is one-to-one. Example 9 The function f(x) is defined by 5 - 2x, x < 1 f: x \mapsto \{ 2 x + 3, x > 1 \text{ Notation This is an example of a piecewisedefined function. It consists of two parts: one linear (for x < 1) and one quadratic (for x > 1). y a Sketch y = f(x), and state the range of f(x). b Solve f(x) = 19. Online x Explore graphs of functions on a given domain using technology. 29 Chapter 2 a Watch out Although the graph jumps at x = 1, the function is still defined for all real values of x: f(0.9) = 5 - 2(0.9) = 3.2 f(1) = (1)2 + 3 = 4 y y = f(x) 5 4 3 O Sketch the graph of y = 5 - 2x for x < 1, and the graph of $y = x^2 + 3$ for x > 1. 1 f(1) lies on the quadratic curve, so use a solid dot on the quadratic curve, and an open dot on the line. x The range is the set of values that y takes and therefore f(x) > 3. by $y = 5 - 2x y 19 y = x^2 + 3$ Note that $f(x) = 19 x^2 = 16 x = \pm 4 x = 4$ The negative solution is where $5 - 2x y 19 y = x^2 + 3$ Note that $f(x) = 19 x^2 = 16 x = \pm 4 x = 4$ The negative solution is where $5 - 2x y 19 y = x^2 + 3$ Note that $f(x) = 19 x^2 = 16 x = \pm 4 x = 4$ The negative solution is where $5 - 2x y 19 y = x^2 + 3$ Note that $f(x) = 19 x^2 = 16 x = \pm 4 x = 4$ The negative solution is where $5 - 2x y 19 y = x^2 + 3 x = 1$ 2x = 19 - 2x = 14 x = -7 Problem-solving Use $x^2 + 3 = 19$ to find the solution in the range x > 1 and use 5 - 2x = 19 to find the solution in the range x > 1. The solutions are x = 4 and x = -7. Exercise 2B 1 For each of the following functions: i draw the mapping diagram ii state if the function is one-to-one or many-to-one iii find the range of the function. a f(x) = 5x - 3, domain $\{x = -3, -2, -1, 0, 1\}$ 4 - 3x 30 Functions and graphs 2 For each of the following mappings: i State whether the mapping is one-to-one, many-to-one or one-to-many-to-one or one-to-one or one-to-many-to-one or one-to-one or one-to-one or one-to-many-to-one or one-to-many-to-one or one-to-many-to-one or one-to-many-to-one or one-to-one or one-to-many-to-one or one-to-one or one-to-many-to-one or one-to-one or one-to-on ii State whether the mapping could represent a function. a y x O d y b y O e x O x y O y c x O y f O x x 3 Calculate the value(s) of a, b, c and d given that: a p(a) = 16 where p: $x \mapsto 3x - 2$, $x \in \mathbb{R}$ b q(b) = 17 where p: $x \mapsto 3x - 2$, $x \in \mathbb{R}$ b q(b) = 17 where p: $x \mapsto 3x - 2$, $x \in \mathbb{R}$ b q(b) = 16 where p: $x \mapsto 3x - 2$, $x \in \mathbb{R}$ b q(b) = 17 where p: $x \mapsto 3x - 2$, $x \in \mathbb{R}$ b q(b) = 16 where p: $x \mapsto 3x - 2$, $x \in
\mathbb{R}$ b q(b) = 16 where p: $x \mapsto 3x - 2$, $x \in \mathbb{R}$ b q(b) = 16 where p: $x \mapsto 3x = 16$ where p: xfunction on a mapping diagram, writing down the elements in the range ii state whether the function is one-to-one or many-to-one. a f(x) = 2x + 1 for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: $x \mapsto \sqrt{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$ b g: the domain {x = -2, -1, 0, 1, 2} Notation Remember, \sqrt{x} means the positive square root of x. 5 For each function: i sketch the graph of y = f(x) ii state the range of f(x) iii c f: x \mapsto 2 sin x for the domain $\{0, -2\}$ e f(x) = ex for the domain $\{x > 0\}$ f $(x) = 7 \log x$, for the domain, $\{x \in \mathbb{R}, x > 0\}$ 6 The following mappings f and g are defined on all the real numbers by 4 - x, x , 4 = 0, x > 4 = 0Chapter 2 P 7 The function s is defined by $x^2 - 6$, x, 0 = x are the values in the value(s) of a such that s(a) = 43. c Solve s(x) = x are the values in the domain that get mapped to themselves in the value(s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value(s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value(s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value(s) of a such that s(a) = 43. c Solve s(x) = x are the value(s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value(s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value (s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value (s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value (s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value (s) of a such that s(a) = 43. c Solve s(x) = x are the values in the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = x are the value (s) of a such that s(a) = 43. c Solve s(x) = 43. c 4 a Sketch y = p(x). (3 marks) b Find the values of a, to 2 decimal places, such that p(a) = 50. (4 marks) The function h has domain -10 < x < 6, and is linear from (-4, 2) to (6, 27). a Sketch y = h(x). (2 marks) Problem-solving b Write down the range of h(x). (1 mark) c Find the values of a, such that h(a) = 12. (4 marks) The function h has domain -10 < x < 6, and is linear from (-4, 2) to (6, 27). a Sketch y = h(x). The graph of y = h(x) will consist of two line segments which meet at (-4, 2). P 10 The function g is defined by g(x) = cx + d where c and d are constants to be found. Given that f(1) = -4 and f(2) = 9. function. The new function is called a composite function. \blacksquare fg(x) means apply g first, then apply f. \blacksquare fg(x) = f(g(x)) g x Watch out f g(x) fg(x) The order in which the functions are combined is important: fg(x) is not necessarily the same as gf(x). fg Example 10 Given f(x) = x2 and g(x) = x + 1, find: a fg(1) 32 b gf(3) c ff(-2) Functions and graphs a fg(1) fg(x) = f(1 + 1) = 22 = 4 g(1) = 1 + 1 f(2) = 22 b gf(3) = g(32) = g(9) = 9 + 1 c ff(-2) = f(-2)2 f(4) = 42 = 16 f(-2) = (-2)2 f(-2) = (-2)2 f(-2) = (-2)2 f(-2) = (that fg(b) = 62. a $fg(x) = f(x2 + 4) = 3(x2 + 4) + 2 = 3x^2 + 14$ g acts on x first, mapping it to $x^2 + 4$. b $gf(x) = g(3x + 2) = (3x + 2)^2 + 4 = 9x^2 + 12x + 8$ Simplify answer. c $f^2(x) = f(3x + 2) = 3(3x + 2) + 2 = 9x + 8$ g acts on the result. d f acts on the result. d f acts on the result. d f acts on the result. f $g(x) = 3x^2 + 14$ If fg(b) = 62 then $3b^2 + 14 = 62$ b $b^2 = 16$ b $b^2 = 46$ b $b^2 = 16$ b $b^2 = 46$ b $b^2 = 16$ b $b^2 = 46$ b $b^2 = 16$ b b^2 = 2(-82) = |x-7| fg(x) = x |x-7| = x y = x y | |7y = |x-7| o 7 x - (x-7) = x - x + 7 = x 2x = 7 x = 3.5 Exercise x+1 g acts on x first, mapping it to 2 f acts on the result. Simplify the answer. Draw a sketch of y = |x-7| and y = x. The sketch shows there is only one solution to the equation |x-7| = x and that it occurs on the reflected part of 2 f acts on the result. the graph. When f(x) < 0, |f(x)| = -f(x). The solution is on the reflected part of the graph so use -(x - 7). This is the x-coordinate at the point of intersection marked on the graph. 2C x 1 Given the functions f(x) = 4x + 1, $g(x) = x^2 - 4$ and h(x) = x, find expressions for the functions: a fg(x) b gf(x) c gh(x) d fh(x) e f 2(x) E B a Find an expression for fg(x). (2 marks) b Solve fg(x) = gf(x). (4 marks) 4 The functions p and q are defined by 1 p(x) = 3x + 4, $x \in \mathbb{R}$ a Find an expression for fg(x). (2 marks) b Solve fg(x) = gf(x). (4 marks) 4 The functions p and q are defined by 1 p(x) = 3x + 4, $x \in \mathbb{R}$ a Find an expression for fg(x). (2 marks) b Solve fg(x) = gf(x). (4 marks) 4 The functions p and q are defined by 1 p(x) = 3x + 4, $x \in \mathbb{R}$ a Find an expression for fg(x). cx + db Solve qp(x) = 16. 34 (3 marks) (3 marks) Functions and graphs E 5 The functions f and g are defined by: f: $x \mapsto |9 - 4x| 3x - 2 g: x \mapsto 2$ a Find fg(6). b Solve fg(x) = x. P (2 marks) (5 marks) 1 6 Given f(x) = x + 2 b Find an expression for f 3(x). 7 The functions s and t are defined by Hint Rearrange the equation in part c s(x) = $2x, x \in \mathbb{R}$ into the form 2x = k where k is a real t(x) = $x + 3, x \in \mathbb{R}$ number, then take natural logs of both a Find an expression for st(x). In a c Solve st(x) = ts(x), leaving your answer in the form 2x = k where k is a real t(x) = $x + 3, x \in \mathbb{R}$ number, then take natural logs of both a Find an expression for st(x). Given f(x) = e 5x and $g(x) = 4 \ln x$, find in its
simplest form: a gf(x) b fg(x) 9 The functions p and q are defined by p: $x \mapsto \ln (x + 3)$, $x \in \mathbb{R}$, x > -3 q: $x \mapsto e 3x - 1$, $x \in \mathbb{R}$ a Find qp(x) and state its range. b Find the value of qp(7). c Solve qp(x) = -126. (2 marks) (2 marks) Hint The range of p will be the set of possible inputs for q in the function qp(7). c Solve qp(7). c Solve qp(x) = -126. (2 marks) (1 mark) (3 marks) 10 The function t is defined by t: $x \mapsto 5 - 2x$ Solve the equation $t_2(x) - (t(x))_2 = 0$. (5 marks) Problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for this problem solving You need to work out the intermediate steps for the problem solving You need to work out the intermediate steps for the problem solving You need to work out the intermediate steps for the problem solving You need to work out the intermediate steps for the problem solving You need to work out the intermediate steps for the problem solving You need to work out the problem solving You ne from (-5, -8) to (0, 12) and from (0, 12) to (14, 5). A sketch of the graph of y = g(x) is shown in the diagram. a Write down the range of g. (1 mark) b Find gg(0). (2 marks) $y_1 - 5$ O y = g(x) 14 x -8 35 Chapter 2 2.4 Inverse function performs the inverse of a function performs the diagram. opposite operation to the original function. It takes the elements in the range of the original function and maps them back into elements of the domain of the original functions. It takes the elements of the original function and maps them back into elements of the domain of the original function. For this reason, inverse functions only exist for one-to-one functions. y = f(x) and y = -6 - 5 - 4 - 3 - 2 - 10 - 1 - 2 - 3 - 4 - 5 - 6 f - 1(x). The range of f(x) is the domain of f(x) is square $\times 2 x x + 7 2 - 7 x 2 2x 2 x 2 - 7 x + 7 x + 7 x 2$ square root $\div 2$ Range of h(x) is h(x) > -7, so domain of x + 7 $= -7 2 \sqrt{An}$ inverse function can often be found using a flow diagram. +7 h-1(x) is x > -7. The range of h(x) is the domain of h-1(x). Example 14 3 Find the inverse of the function f(x) = $-7 2 \sqrt{An}$ inverse function can often be found using a flow diagram. +7 h-1(x) is $x > -7 2 \sqrt{An}$ inverse function f(x) = $-7 2 \sqrt{An}$ inverse function f(x) = -7 2 $\neq 1\}$ by changing the subject of the formula. x-1 Let y = f(x) 3 y = x-1 y(x-1) = 3 yx - y = 3 yx = 3 + y 3 + y x = y Range of f(x) is $f(x) \neq 0$, so domain of f-1(x) is $x \neq 0.3 + x$ Therefore f-1(x) = x, $x \neq 0.3 3 = f(4) = 14 - 133 + 144 = 14 - 143 + 144 = 14 - 143 + 144 = 14 - 143 + 144 = 14 + 144 = 14 + 144 = 14 + 144 = 14 + 144 = 14 + 144 = 14 + 144 = 14 + 144 = 14 + 144 = 144 = 144 + 144 = 144 + 144 = 144 + 144$ y = f(x). Rearrange to make x the subject of the formula. Define f-1(x) in terms of x. Check to see that at least one element works. Try 4. Note that f-1f(4) = 4. f(x) 4 1 f -1(x) Functions and graphs Example $\overline{15}$ The function, $\overline{f(x)} = \sqrt{x-2}$, $x \in \mathbb{R}$, x > 2. a State the range of f(x). b Find the function f-1(x) and state its domain and range. c Sketch y = f(x) and y = f-1(x) and the line y = x. a The range of f(x) is $y \in \mathbb{R}$, y > 0. b c _____ $y = \sqrt{x - 2} y^2 = x - 2 x^2 = y - 2 y = x^2 + 2$ The inverse function is $f - 1(x) = x^2 + 2$. The domain of $f - 1(x) = x^2 + 2$. The domain of $f - 1(x) = x^2 + 2$. The domain of $f - 1(x) = x^2 + 2$. The domain of $f - 1(x) = x^2 + 2$. The domain of $f - 1(x) = x^2 + 2$. The domain of $f - 1(x) = x^2 + 2$. the range is f(x) > 0, or y > 0. Interchange x and y. Always write your function in terms of x. The range of f(x) is the same as the domain of f(x). The graph of f(x) in the line y = x. This is because the reflection transforms y to x and x to y. y = f(x) = x - 20123456 xExample 16 The function f(x) is defined by f(x) = x 2 - 3, $x \in \mathbb{R}$, x > 0. a Find f-1(x). b Sketch y = f-1(x) and state its domain. a Let y = f(x) = y + 3 = x y + 3 = x + 3 = x + 3 = x + 3inverses using technology. First sketch f(x). Then reflect f(x) in the line y = x. x - 3 The range of the original function is f(x) > -3. The domain of f-1(x) is $x \in \mathbb{R}$, x > -3. 37 Chapter 2 c When $f(x) = x \cdot 2 - 3 = x \cdot 2 - x - 3 = 0 \sqrt{13} \cdot 1 + So \cdot x = 2$. 2 Exercise Problem-solving y = f(x) and y = f-1(x) intersect on the line y = x. This means that the solution to f(x) = f - 1(x) is the same as the solution to f(x) = x. From the graph you can see that the solution must be positive, so ignore the negative solution to the equation. 2D 1 For each of the following functions f(x): i state the range of f(x) ii determine the equation of the inverse function f - 1(x) iii state the domain and range of f-1(x) iv sketch the graphs of y = f(x) and y = f-1(x) on the same set of axes. x+5 a f: $x \mapsto 2x + 3$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ a f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 -
7$, $x \in \mathbb{R}$ b f: $x \mapsto x^3 - 7$, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7, $x \in \mathbb{R}$ b f: x \mapsto x^3 - 7 function is self-inverse if f-1(x) = f(x). In this case ff(x) = x. 3 Explain why the function g(x) ii determine the equation of the inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the following functions g(x) is not identical to its inverse. 4 For each of the followin graphs of y = g(x) and y = g-1(x) on the same set of axes. 1 a $g(x) = x^2 - 1$, $\{x \in \mathbb{R}, x > 0\}$ x, $\{x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$, $x \in \mathbb{R}, x > 2\}$ for $(x) = x^2 - 6x + 5$. complete the square for the function t(x). (5 marks) 6 The function m(x) is defined by $m(x) = x^2 + 4x + 9$, $x \in \mathbb{R}$, x > a, for some constant a. a State the least value of a for which m-1(x). (3 marks) b Determine the equation of m-1(x). (5 marks) b Determine the equation of m-1(x). (7 marks) b Determine the equation of m-1(x). (8 marks) b Determine the equation of m-1(x). (9 marks) b Dete $x \in \mathbb{R}, x \neq 2$. x-2 a What happens to the function as x approaches 2? b Find h-1(3). c Find h-1(x), stating clearly its domain. d Find the elements of the domain that get mapped to themselves by the functions m and n are defined by m: $x \mapsto 2x + 3$, $x \in \mathbb{R}, x-3$ n: $x \mapsto 2$, $x \in \mathbb{R}, x = 3$ n: $x \mapsto 2$, $x \in \mathbb{R}, x = 3$ n: $x \mapsto 2$. about the functions m and n? P E/P E 9 The functions s and t are defined by $3 s(x) = 1, x \neq -1, x+1, 3-x, t(x) = 1, x \neq -1, x+1, x \neq -1, x \neq$ The functions f and g are defined by f: $x \mapsto ex - 5$, $x \in \mathbb{R}$ g: $x \mapsto \ln(x - 4)$, x > 4 a State the range of f. b Find f-1, the inverse function of f, stating its domain. (3 marks) c On the same axes, sketch the curves with equation y = f(x) and y = f-1(x), giving the coordinates of all the points where the curves cross the axes. (4 marks) d Find (3 marks) g-1, the inverse function of g, stating its domain. e Solve the equation g-1(x) = 11, giving your answer to 2 decimal places. E/P (1 mark) 12 The function f is defined by 3(x + 2) 2 -, x > 4 f: $x \mapsto$, x > 4 f: $x \mapsto$, x > 4 f: $x \mapsto$, x > 4 f: $x \mapsto$ (4 mark) (4 marks) (2 marks) (4 marks) 39 Chapter 2 2.5 y = |f(x)| = To sketch the graph of y = $f(|x|) \equiv To$ sketch the graph of y = $f(|x|) \equiv To$ sketch the graph of y = $f(|x|) \equiv To$ sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch the graph of y = f(|x|) = To sketch graph of y = f(x) for x > 0. • Reflect this in the y-axis. $O -3 \ 3x -6$ Example 17 $f(x) = x^2 - 3x - 10$ a Sketch the graph of y = f(x). b Sketch the graph of y = f(x). c Sketch the graph of y = f(5. y The graph cuts the y-axis at -10. y = f(x) This is the sketch of $y = x^2 - 3x - 10$. -2 O 5 x The sketch includes the points where the graph intercepts the coordinate axes. A sketch does not have to be to scale. $-10 y b y = |f(x)| = |x^2 y 10 - 2 40 O - 3x - 10|$ Online Explore graphs of modulus functions using technology. y = |f(x)| 5 x Reflect the part of the curve where y = f(x) < 0 (the negative values of y) in the x-axis. x Functions and graphs c $y = f(|x|) = |x|^2 - 3|x| - 10$ y y = f(|x|) Reflect the part of the curve where x > 0 (the positive values of x) in the y-axis. O $-5 \times 5 - 10$ Example 18 g(x) = sin x, $-360^\circ < x < 360^\circ$ a Sketch the graph of y = g(x). c Sketch the graph of y = f(|x|) and
y = f(|x|) reflect the part of the curve where x > 0 (the positive values of x) in the y-axis. O $-5 \times 5 - 10$ Example 18 g(x) = sin x, $-360^\circ < x < 360^\circ$ a Sketch the graph of y = g(x). graph of y = g(|x|). a y The graph is periodic and passes through the origin, (±180, 0) and (±360, 0). $y = \sin x 1 \leftarrow \text{Year } 1$, Section 9.5 - 360 - 180 O 360 x 180 - 1 b y y = $|\sin x| 1 - 360 - 180$ Reflect the part of the curve below the x-axis in the x-axis. O 180 360 x - 1 c y 1 - 360 - 180 O y = $\sin|x| 180 360$ Reflect the part of the curve where x > 0 in the y-axis. x - 1 41 Chapter 2 Example 19 y The diagram shows the graph of y = h(x), with five points labelled. Sketch each of the following graphs, labelling the points of intersection with the coordinate axes. a y = |h(x)| B (-2.5, 15) 11 C A y = h(x) D - 7 O x 3 E (6, -5) b y = h(|x|) a y The parts of the curve below the x-axis are reflected in the x-axis. B (-2.5, 15) 11 A C E 9 (6, 5) O -7 The point E was reflected, so the new coordinates are E'(6, 5). x 3 b The points A, B, C and D are unchanged. y = |h(x)| D y The part of the curve to the right of the y-axis is reflected in the y-axis. B (-2.5, 15) 11 A C E 9 (6, -5) The old points A and B had negative x-values so they are no longer part of the graph. x The points C, D and E are unchanged. There is a new point of intersection with the x-axis at (-3, 0). 2E 1 $f(x) = x^2 - 7x - 8$ a Sketch the graph of y = f(x). b Sketch the graph the graph of y = |g(x)|. c Sketch the graph of y = g(|x|). 3 h: $x \mapsto x$ (x - 1)(x - 2)(x + 3) a Sketch the graph of y = h(x). c Sketch the graph of y = h(x). Functions and graphs P a 4 The function k is defined by k(x) = 2, a > 0, $x \in \mathbb{R}$, $x \neq 0$. x a Sketch the graph of y = k(x). b Explain why it is not necessary to sketch y = |k(x)| and y = k(|x|). a The function m is defined by $m(x) = (x \in \mathbb{R}, x \neq 0, x \in$ labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes. a y = |p(x)| (3 marks) y = p(x) 3D A -8 C -2 O E (2, 1) x B (-4, -5) y E 6 Sketch each of the following graphs, labelling the points corresponding to A, B, C, D and E, and any points of intersection with the coordinate axes. 7 y = q(x) The diagram shows the graph of y = q(x) with 7 points labelled. a y = |q(x)| (4 marks) b y = q(|x|) (3 marks) D (- 4, 3) A -10 a k(x) = __x, a > 0, x \neq 0 D C -5 -3 O G 4 x -4 F B (- 8, - 9) a Sketch the graph of y = k(x). b Sketch the graph of y = k(x). c Sketch the graph of y = k(x). B a m(x) = __x, a < 0, x \neq 0 a Sketch the graph of y = m(x). b Describe the relationship between y = |m(x)| and y = m(|x|). 9 f(x) = ex and g(x) = e-x a Sketch the graphs of y = f(x) and y = g(x). c Sketch the graphs of y = f(|x|) and y = g(|x|) on the same axes. 43 Chapter 2 E/P 10 The function f(x) is defined by -2x - 6, -5 < xy = 1, $y = (x + 1)^2$, $y = (x + 1)^2$, y = 1 < x < 2 a Sketch f(x) stating its range. Problem-solving { A piecewise function like this does not have to be continuous. Work out the value of both expressions when x = -1 to help you with your sketch. (5 marks) b Sketch the graph of y = f(x). (3 marks) c Sketch the graph of y = f(x). (3 marks) 2.6 Combining transformations You can use combinations of the following transformations of a function to sketch graphs of more complicated transformations. $-a \equiv f(x + a)$ is a translation by the vector (0) $\equiv a(x)$ is a vertical stretch of scale factor 1 $a \equiv f(x)$ in the y-axis. Links You can think of f(-x) and -f(x) as stretches with scale factor -1. \leftarrow Year 1, Sections 4.6, 4.7 \blacksquare -f(x) reflects f(x) in the x-axis. y Example 20 B (6, 4) The diagram shows a sketch of the graphs of: O a y = 2f(x) - 1 b y = f(x + 2) + 12 A (2, -1) 1 c y = f(2x) d y = -f(x - 1) 4 In each case, find the coordinates of the points O, A and B are (0, -1), (2, -3) and (6, 7) respectively. 44 x Next apply the translation. The solid curve is the graph of y = 2f(x) - 1, as required. This is a 0 translation of y = 2f(x) - 2. x Functions and graphs by y = f(x + 2) + 2 y Apply the translation inside the brackets first. The dotted curve is the graph of y = f(x + 2), 2 which is a translation of y = f(x + 2), by 0 vector (). 2 The images of O A and B are (-2, 2), (0, 1) and (4, 6) respectively. 1 c y = $_1(2x)$, 4 y (3, 4) Apply the stretch inside the brackets first. The dotted curve is the graph of y = $_1(2x)$, which is a horizontal stretch of y = f(2x), which is a horizontal stretch of y = f(2x). with scale factor _14 (3, 1) (1, - 0.25) x (0, 0) (1, -1) The images of O, A and B are (0, 0), (1, -0.25) and (3, 1) respectively. d y = -f(x - 1) y (7, 4) (1, 0) Apply the translation inside the brackets first. The dotted curve is the graph of y = f(x - 1) y (7, 4) (1, 0) Apply the translation of y = f(x - 1) y (7, 4) (1, 0) Apply the translation of y = f(x - 1) y (7, 4) (1, 0) Apply the translation of y = f(x - 1) y (7, 4) (1, 0) Apply the translation inside the brackets first. the brackets. The solid curve is the graph of y = -f(x - 1), as required. This is a reflection of y = f(x - 1) in the x-axis. The images of O, A and B are (1, 0), (3, 1) and (7, -4) respectively. 45 Chapter 2 Example 21 $f(x) = \ln x$, x > 0 Sketch the graphs of a y = 2f(x) - 3 b y = |f(-x)| Show, on each diagram, the point where the graph meets or crosses the xaxis. In each case, state the equation of the asymptote. y a y Online x=0 y = In x Explore combinations of transformations using technology. Problem-solving O x 1 2ln x - 3 = 0 3 ln x = 2 3 x = e2 = 4.48 (3 s.f.) The graph y = 2ln x - 3 will cross the x-axis at (4.48, 0). y O x=0 You have not been asked to sketch y = f(x) in this question, but it is a good idea to do this before sketching transformations of this graph. Sketch y = f(x), labelling its asymptote and the coordinates of the point where it crosses the x-axis. \leftarrow Year 1, Section 14.3 Solve this equation to find the x-intercept of y = 2f(x) - 3. 2 In $x - 3 \times 4.48$ The original graph underwent a vertical stretch by a scale factor of 2 and then a vertical translation 0 by vector (). -3 b The graph of y = f(-x) is a reflection of y = f(x) in the y-axis. y = In (-x) x=0 y = In x The original graph is first reflected in the y-axis. The x-intercept becomes (-1, 0). -1 O 46 1 x The asymptote is unchanged. x Functions and graphs y y = In (-x) |-1 O Exercise x=0 To sketch the graph of y = |f(-x)| reflect any of y = |f(-x)| reflect any of y = |f(-x)| reflect any of y = In x The original graph is first reflected in the y-axis. The x-intercept becomes (-1, 0). -1 O 46 1 x The asymptote is unchanged. x Functions and graphs y y = |In (-x)| -1 O Exercise x=0 To sketch the graph of y = |f(-x)| reflect any of y = |f(-x)| reflect any of y = |In (-x)| reflect any of y = In (-x) + In (-x) negative y-values of y = f(-x) in the x-axis. x 2F 1 The diagram shows a sketch of the graph y = f(x). The curve passes through the origin O, the point A(-2, -2) and the point B(3, 4). On separate axes, sketch the graphs of: a y = 3f(x) + 21 c y = (f(x + 1) 2 e y = |f(x)| y B(3, 4) b y = f(x - 2) - 5 d y = -f(2x) In each case find the coordinates of the images of the points O, A and B. x O f y = |f(-x)| y = f(x) A(-2, -2) 2 The diagram shows a sketch of the graph y = f(x). The curve has a maximum at the point A(-1, 4) and crosses the axes at the points (0, 3) and (-2, 0). 1 1 a y = $3f(x - 2) b y = _{-}f(x) + 4 d y = -2f(x + 1) y A(-1, 4) 3 y = f(x) - 2 e y = 2f(|x|) x O$ For each graph, find, where possible, the coordinates of the maximum or minimum and the coordinates of the intersection points with the axes. 3 The diagram shows a sketch of the graph y = f(x). The lines x = 2 and y = 0 (the x-axis) are asymptotes to the curve. On separate axes, sketch the graphs of: a y = 3f(x) - 1 b y = f(x + 2) + 4 c y = -f(2x) d y = f(|x|) For each part, state the equations of the asymptotes and the new coordinates of the point A. y 1 O y = f(x) A 2 x 47 Chapter 2 E 4 The function g is defined by g: $x \mapsto (x - 2)2 - 9$, $x \in \mathbb{R}$. a Draw a sketch of the graph of y = g(x), labelling the turning points and the x- and y-intercepts. (3 marks) b Write down the coordinates of the turning point when the curve is transformed as follows: i 2g(x - 4) (2 marks) ii g(2x) (2 marks) c Sketch the curve with equation y = g(|x|). On your sketch show the coordinates of all turning points and all x- and y-intercepts. (4 marks) 5 $h(x) = 2 \sin x$, $-180^{\circ} < x < 180^{\circ}$. a Sketch the graph of y = h(x). b Write down the coordinates of the minimum, A, and the maximum, B. c Sketch the graphs of: 1 1 1 ii _h(_x) iii _|h(-x)| 4 2 2 In each case find the coordinates of the points O, A and B. i h(x - 90°) + 1 2.7 Solving modulus problems You can use combinations together with |f(x)| and f(|x|) and an understanding of domain and range to solve problems. Example 22 Given the function t(x) = 3|x - 1| - 2, $x \in \mathbb{R}$, a sketch the graph of y = |x|. Use transformations to sketch the graph of y = |x|. Use transformation by vector (). 0 1 O x 1 y y = 3|x - 1| 3 Step 2 Vertical stretch, scale factor 3. O x 1 y y = 3|x - 1| - 2 Step 3 0 Vertical translation by vector (). -2 1 x O (1, -2) b The range of the function t(x) is $y \in \mathbb{R}$, y > -2. The graph has a minimum at (1, -2). y c B First draw a sketch of y = 3|x - 1| - 2 and the line 1 y = x + 3. 2 A 3 1 x O (1, -2) The sketch shows there are two solutions, at A and B, the points of intersection. 49 Chapter 21 At A, $3(x - 1) - 2 = x + 321 3x - 5 = x + 325 x = 8216 x = 51 \text{ At B}, -3(x - 1) - 2 = x + 327 - x = 224 x = - 7416 \text{ The solutions are } x = _ and x = - _ 7416 \text{ The solution on the original part of the graph. When } f(x) < 0$, |f(x)| = -f(x), so use (3x - 1) - 2 to find the solution on the reflected part of the graph. This
is the solution corresponding to point B on the sketch. Example 23 y The function is shown in the diagram. a State the range of f. b Give a reason why f - 1 does not exist. -6 c Solve the inequality f(x) > 5. a The range of f(x) is f(x) < 6. y = f(x) b f(x) is a many-to-one function. Therefore, f - 1 does not exist. c f(x) = 5 at the points A and B. f(x) > 5 between the points A and B. y = 6 A The greatest value f(x) can take is 6. For example, f(0) = f(-6) = 0. Problem-solving 5 Only one-to-one functions have inverses. x O y = f(x) At A, 6 - 2(x + 3) = 5 - 2(x-11x + 3 = 25x = -250x O Add the line y = 5 to the graph of y = f(x). Between the points A and B, the graph of y = f(x) is above the line y = 5. This is the solution on the original part of the graph. Functions and graphs At B, 6 - (-2(x + 3)) = 52(x + 3) = -11x + 3 = -27x = -< - 22 Exercise P When f(x) < 0, |f(x)| = -f(x), so use the negative argument, -2(x + 3). This is the solution on the reflected part of the graph of y = f(x) ii state the range of the function. a f: $x \mapsto 4 |x| - 3$, $x \in \mathbb{R}$ 1 b f(x) = -1, $x \in \mathbb{R}$ 3 c f(x) = -2 |x|-3|x| + 6, $x \in \mathbb{R}$, a sketch the graph of y = q(x) b shade the region of the graph that satisfies y < q(x). 4 The function f is defined as f: $x \mapsto 4|x + 6| + 1$, $x \in \mathbb{R}$. a Sketch the graph of y = g(x) b state the range of the function f is defined as f: $x \mapsto 4|x + 6| + 1$, $x \in \mathbb{R}$, a sketch the graph of y = g(x) b state the range of the function f is defined as f: $x \mapsto 4|x + 6| + 1$, $x \in \mathbb{R}$, a sketch the graph of y = g(x) b state the range of the function. range of the function c solve the equation g(x) = x + 1. 51 Chapter 2 E/P 6 The functions m and n are defined as Problem-solving m(x) = -2x + k, $x \in \mathbb{R}$ where k is a constant. The equation m(x) = n(x) has no real roots. Find the range of possible values for the constant k. E/P (4 marks) 7 The functions s and t are defined as s(x) = -10 - x, $x \in \mathbb{R}$ where b is a constant. (4 marks) The equation s(x) = t(x) has exactly one real root. Find the value of b. E/P 8 The function h is defined by 2 h(x) = -10 - x, $x \in \mathbb{R}$ where b is a constant. (4 marks) The equation s(x) = t(x) has exactly one real root. the range of h. b Give a reason why h-1 y (1 mark) does not exist. c Solve the inequality h(x) < -6. O (1 mark) (4 marks) y = h(x) d State the range of values of k for which the 2 equation h(x) = x + k has no solutions. (4 marks) y = h(x) d State the range of values of k for which the 2 equation h(x) = x + k has no solutions. (4 marks) 3 E/P 9 The diagram shows a sketch of part of the graph y = h(x), where h(x) = a - 2|x + 3|, $x \in \mathbb{R}$. The graph intercepts the y-axis at (0, 4). a Find the value of a. E/P P Q 3 (3 marks) (5 marks) y = h(x) 52 y (1 mark) b Solve the equation m(x) = x + 2. (4 marks) 5 Given that m(x) = x + 2. (4 marks) 5 Given that m(x) = x + 2. (4 marks) 5 Given that m(x) = x + 2. (4 marks) y = h(x) 52 y (1 mark) b Solve the equation m(x) = x + 2. (4 marks) $x \in \mathbb{R}$. a constant, has two distinct roots c state the set of possible values for k. x O 10 The diagram shows a sketch of part of the graph y = m(x), where m(x) = -4|x + 3| + 7, $x \in \mathbb{R}$. a State the range of m. y 4 (2 marks) b Find the coordinates of P and Q. 1 c Solve h(x) = x + 6. 3 x (4 marks) x O -5 y = m(x) Functions and graphs of y = f(x) and y = g(x). y y = f(x) and y = g(x). y y = f(x) and y = g(x) B a Find the coordinates of the points A and B. b Find the area of the region R. 2 The functions f and g are defined as: y = g(x) f(x) = -|x - 3| + 2, $x \in \mathbb{R}$ 64 Show that the area of the shaded region is 3y = f(x) O Mixed exercise x = 2 + 1. b Hence, or otherwise, find the area of the shaded region is 3y = f(x) O Mixed exercise x = 1 a On the same axes, sketch the graphs of y = 2 - x and y = 2|x - 3| + 2, $x \in \mathbb{R}$ 64 Show that the area of the shaded region is 3y = f(x) O Mixed exercise x = 1 a On the same axes, sketch the graphs of y = 2 - x and y = 2|x - 3| + 2. values of x for which 2 - x = 2|x + 1|. E/P 1 2 The equation |2x - 11| = x + k has exactly two distinct solutions. 2 Find the range of possible values of k. (4 marks) E/P 4 a On the same set of axes, sketch y = |12 - 5x| and y = -2x + 3. (3 marks) b State with a reason whether there are any solutions to the equation |12 - 5x| = -2x + 3 (2 marks) 53 Chapter 2 5 For each of the following mappings: i state whether the mapping is one-to-one, many-to-one or one-to-many ii state whether the mapping could represent a function. y a x O y d O e x O E y b c x y O O f x y x y O x 6 The function f(x) is defined by -x, x < 1 f(x) = { x - 2, x . 1 a Sketch the graph (3 marks) b Determine y = (3 marks) c Sketch y = two graphs. g-1(x) E stating its range. on the same axes as y = g(x), stating the relationship between the 9 The function f is defined by 2x + 3 f: $x \mapsto (x \in \mathbb{R}, x > 1)$ for x = 1 - 1 a Find f (x). b Find: i the range of (4 marks) f - 1(x) ii the domain of f - 1(x) 54 (2 marks) (2 marks) Functions and = 0, find (4 marks) f -1, stating its domain. 13 The functions f and g are given by $f: x \mapsto 4x - 1$, { $x \in \mathbb{R}$ } 3 1 g : $x \mapsto 2x - 1$ Find in its simplest form: E a the inverse function f - 1 (2 marks) b the composite function gf, stating its domain (3 marks) c the values of x for which 2f(x) = g(x), giving your answers to 3 decimal places. (4 marks) 14 The functions f and g are given by x f : x \mapsto x, a Find an expression for f -1(x). (2 marks) d Use algebra to find the values of x for which g(x) = f(x) + 4. (4 marks) 15 The function n(x) is defined by 5 - x, x < 0 n(x) $= \{x^2, x, 0 \text{ a Find } n(-3) \text{ and } n(3) \text{ b Solve the equation } n(x) = 50.55 \text{ Chapter } 2 \text{ 16 } g(x) = \tan x, -180^\circ < x < 180^\circ \text{ a Sketch the graph of } y = g(x) \text{ b Sketch the graph of } y = g(x$ separate diagrams, the graphs of a y = f(2x) + 1 (3 marks) b y = |f(x)| (3 marks) b y = |f(x)| (3 marks) b y = -f(x - 2) (3 marks) c y = -f(x - 2) (3 marks) b y = |f(x)| (Sketch the graph of y = |f(x)| and hence find the values of x for which |f(x)| = 2. (4 marks) 19 The function f is defined by $f: x \mapsto |2x - a|$, $\{x \in \mathbb{R}\}$, where a is a positive constant. a Sketch the graph of y = f(x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph of y = f(2x), showing the
coordinates of the points where the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph cuts the axes. (3 marks) b On a separate diagram, sketch the graph cuts the axes. (3 marks) b On a separate diagram, sket the coordinates of the points where the graph cuts the axes. (2 marks) 1 c Given that a solution of the equation f(x) = x is x = 4, find the two possible 2 values of a. (4 marks) E/P 20 a Sketch the graph meets the axes. (3 marks) 1 (4 marks) b Using algebra solve, for x in terms of a, |x - 2a| = x. 3 c On a separate diagram, sketch the graph of y = a - |x - 2a|, where a is a positive constant. Show the coordinates of the points where the graph meets the coordinate axes. (3 marks) 1 (2 marks) b On the same axes, sketch the graph of y = x c Explain how your graphs show that there is only one solution of the equation <math>x/2x + a = -1 = 0. (2 marks) functions and graphs E/P 22 The diagram shows part of the curve with equation y = f(x), where $f(x) = x^2 - 7x + 5 \ln x + 8$, x > 0 The points A and B are the stationary points of the curve with equation y = -3f(x - 2). (3 marks) c Find the coordinates of the stationary points of the curve with equation y = -3f(x - 2). -2). State, without proof, which point is a maximum and which point is a minimum. (3 marks) E/P B x O y 23 The function f has domain -5 < x < 7 and is linear from (-3, -2) to (7, 18). The diagram shows a sketch of the function. a Write down the range of f. (1 mark) b Find ff(-3). (7, 18) y = f(x) (-5, 6) (2 marks) c Sketch the graph of y = |f(x)|, marking the points at which the graph meets or cuts the axes. (3 marks) (-3, -2) x O The function p is defined by $y_p: x \mapsto -2|x + 4| + 10$ The diagram shows a sketch of the graph. a State the range of p. b Give a reason why p -1 (1 mark) does not exist. c Solve the inequality p(x) > -4. (1 mark) (4 marks) d State the range of values of k for which the equation 1 p(x) = -x + k has no solutions. (4 marks) 2 2 x O y = p(x) Challenge a Sketch, on a single diagram, the graphs of $y = a^2 - x^2$ and y = |x + a|, where a is a constant and a > 1. b Write down the coordinates of the points where the graph of $y = a^2 - x^2$ cuts the coordinate axes. c Given that the two graphs intersect at x = 4, calculate the value of a. 57 Chapter 2 Summary of key points 1 A modulus function is, in general, a function of the type y = |f(x)| = -f(x) + b|, sketch y = ax+ b then reflect the section of the graph below the x-axis in the x-axis. 3 A mapping is a function fg(x) = f(g(x)) g x f g(x) fg(x) fg(are inverses of each other. ff -1(x) = x and f -1(x) = x. 6 The graphs of y = f(x) are reflections of each another in the line y = x. 7 The domain of f(-1)(x) = x. 7 The domain of axis) in the x-axis \bullet Delete the parts below the x-axis 10 To sketch the graph of y = f(x) of x > 0 \bullet Reflect this in the y-axis 11 f(x + a) is a horizontal stretch of scale factor a 14 af(x) is a vertical stretch of scale factor a. 15 f(-x)reflects f(x) in the y-axis. 16 -f(x) reflects f(x) in the x-axis. 58 B 3 Sequences and series Objectives After completing this chapter you should be able to: \bullet Find the nth term of a geometric sequence \rightarrow pages 66-70 \oplus Prove and use the formula for the sum of a finite geometric series \rightarrow pages 70-73 \oplus Prove and use the formula for the sum to infinity of a convergent geometric series \rightarrow pages 73-76 \oplus Use sigma notation to describe series \rightarrow pages 76-78 \oplus Generate sequences from recurrence relations \rightarrow pages 79-83 \oplus Model real-life situations with sequences and series \rightarrow pages 83-86 Prior knowledge check 1 Write down the next three terms of each sequences and series can be found in nature, and can be used to model population growth or decline, or the spread of a virus. \rightarrow Exercise 31, Q12 2 Solve, giving your answers to 3 s.f.: a 2x = 50 b 0.2x = 0.0035 c 4 × 3x = 78 732 \leftarrow Year 1, Section 14.6 59 Chapter 3 3.1 Arithmetic sequences II nan arithmetic sequences II nan arithmetic sequences II nan arithmetic sequences are consecutive terms is constant. 5, 7, +2 9, +2 11, +2 12.5, 10, 7.5, 5, -2.5 -2.5 4, 7, +3 12, +5 19, +7 This sequence is arithmetic. The difference between consecutive terms is -2.5. The sequence is arithmetic. The difference between consecutive terms is -2.5. The sequence is arithmetic. The difference between consecutive terms is -2.5. The sequence is arithmetic. The difference between consecutive terms is -2.5. The sequence is arithmetic. sequence is not arithmetic. \blacksquare The formula for the nth term \bullet a is the first term and term a a Write down the first 5 terms of the sequence. b Find the 99th term in the sequence c Find the first term in the sequence that is negative. a un = 55 - 2(1) = 53 n = $2 \rightarrow u^2 = 55 - 2(2) = 51$ n = $3 \rightarrow u^3 = 55 - 2(3) = 49$ n = $4 \rightarrow u^4 = 55 - 2(4) = 47$ n = $5 \rightarrow u^5 = 55 - 2(5) = 45$ b $u^{99} = 55 - 2(99) = -143$ c 55 - 2n < 0 - 2n < 0-55 n > 27.5 n = 28 u 28 = 55 - 2(28) = -1.60 Online Use the table function on your calculator to generate terms in the sequence for this function, or to check an nth term. Remember, n is the position in the sequence for this function, or to check an nth term. Remember, n is the position in the sequence for this function, or to check an nth term. find the first negative term, set un < 0 and solve the inequality. n is the term number so it must be a positive integer. Sequences and series Example 2 Find the nth term of each arithmetic sequence. a 6, 20, 34, 48, 62 b 101, 94, 87, 80, 73 a a = 6, d = 14 un = 6 + 14(n - 1) u the values of a and d into the formula a + (n - 1)d and simplify. b = 101 - 7n + 7 un = 100 - 7n (1) u 8 = 20, so 8a + b = 20. (2) (2) - (1) gives: 5a = 15 a = 3 Substitute a = 3 in (1): 9+b=5 b = -4 Constants are a = 3 and b = -4. Exercise If the sequence is decreasing then d is negative. Problem-solving You know two terms and there are two unknowns in the expression for the nth term. You can use this information to form two simultaneous equations. \leftarrow Year 1 Section 3.1 Substitute n = 3 and $u^3 = 5$ in un = an + b. Solve simultaneously. 3A 1 For each sequence ii write down a and d. a un = 5n + 2b un = 9 - 2nc un = 7 + 0.5n d un = n - 10 61 Chapter 3 2 Find the nth terms and the 10th terms in the following arithmetic progressions: P a 5, 7, 9, 11, ..., b 5, 8, 11, 14, ..., c 24, 21, 18, 15, ..., d -1, 3, 7, 11, ..., e x, 2x, 3x, 4x, ..., f a, a + d, a + 2d, a $5x, \dots, 35x$ Find an expression for u n and set it equal to the final term in the sequence. Solve the equation to find the value of n. f a, a + d, a + 2d, \dots, a + (n - 1)d P 4 The first term of an arithmetic sequence is 14. The fourth term is 32. Find the common difference. P 5 A sequence is generated by the formula un = pn + q where p and q are constants to be found. Given that u6 = 9 and u9 = 11, find the constants p and q. P 6 For an arithmetic sequence are 5p, 20 and 3p, where p and u9 = 9. Find the value of the 10th term is -6. Find the value of the 10th term is -6. Find the first negative term in the sequence are 5p, 20 and 3p, where p and u9 = 9. Find the first negative term is -6. Find the value of the 10th term is -6. Find the value of the 10th term is -6. Find the value of the 10th term is -6. Find the value of the 10th term is -6. Find the value of the 10th term is -6. is a constant. Find the 20th term in the sequence is 41. Find the value of k, giving your answer in the form p + q / 5, where p and g are integers to be found. (4 marks) Challenge The nth term of an arithmetic sequence is $u n = \ln a + (n - 1) \ln b$ where a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. Find the values of a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. Find the values of a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. Find the values of a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. Find the values of a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. Find the values of a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. Find the values of a and b are integers. $u^3 = \ln 16$ and $u^7 = \ln 256$. arithmetic series is the sum of the first n terms of a series. Example 4 Prove that the sum of the first 100 natural numbers is 5050. S100 = +12 + 3 + ... + 98 + 99 + 100 S100 = 100 + 99 + 98 + ... + 3+ 2 + 1 (1) (2) Adding (1) and (2): $2 \times S100 = 100 \times 101 \ 100 \times 101 \ S100 =$ 2 = 5050 The natural numbers are the positive integers: 1, 2, 3, 4, ... Problem-solving Write out the sum longhand, then write it out in reverse. You can pair up the numbers so that each pair has a sum of 101. There are 100 pairs in total. The sum of the first n terms of an arithmetic series is given by the formula as n Sn = (a + 1) 2 where a is the first term and d is the common difference. You can also write this formula as n Sn = (a + 1) 2 where a is the first term. Example 5 n Prove that the sum of the first n terms of an arithmetic series is (2a + (n - 1)d) 2 Sn = a + (a + d) + (a + 2d) + ... + (a + 2d) + ..(a + (n - 2)d) + (a + (n - 1)d) (1) Sn = (a + (n - 1)d) + (a + (n - 2)d) + ... + (a + 2d) + (a + d) + a (2) Write out the terms of the sum. This is the sum reversed. Adding (1) and (2): 2 × Sn = n(2a + (n - 1)d) + (a + d) + a (2) Write out the terms of the sum. This is the sum reversed. Adding together the two sums. Problem-solving You need to learn this proof for your exam. 63 Chapter 3 Example 6 Find the sum of the first 50 terms of the arithmetic series 32 + 27 + 22 + 17 + 12 + ... a = 32, d = -5 Write down a and d. 50 S50 = (2(32) + (50 - 1)(-5)) 2 Substitute into the formula. S50 = -4525 Simplify. Example 7 Find the least
number of terms required for the sum of 4 + 9 + 14 + 19 + ... to exceed 2000. Always establish what you are given in a question. As you are adding on positive terms, it is easier to solve the equality Sn = 2000.4 + 9 + 14 + 19 + ... > 2000 n Using Sn = (2a + (n - 1)d) 2 n 2000 = n(8 + 5n - 5) Knowing a = 4, d = 5 and Sn = 2000, you need to find n. n Substitute into <math>Sn = (2a + (n - 1)d) 2 n 2000 = n(5n + 3) 4000= 5n2 + 3n 0 = 5n2 + 3n - 4000Solve using the quadratic formula. $-3 \pm \sqrt{9} + 80\ 000\ n =$ 27 + 24 + 21 + ... (40 terms) d 5 + 1 + -3 + -7 + ... (14 terms) e 5 + 7 + 9 + ... + 75 f 4 + 7 + 10 + ... + 91 g 34 + 29 + 24 + 19 + ... + -111 h (x + 1) + (2x + 1) + (3x + 1) + ... + (21x + 1) Hint For parts e to h, start by using the last term to work out the number of terms in the series. 2 Find how many terms of the following series are needed to make the given sums, a 5 + 8 + 11 + 14 + ... = 670 Hint Set the expression for S n egual to the b 3 + 8 + 13 + 18 + ... = 1575 total and solve the resulting eguation to find n. c 64 + 62 + 60 + ... = 0 d 34 + 30 + 26 + 22 + ... = 112 64 Seguences and series P 3 Find the sum of the first 50 even numbers. P 4 Find the least number of terms for the sum of 7 + 12 + 17 + 22 + 27 + ... to exceed 1000. P 5 The first term of an arithmetic series is 4. The sum to 20 terms is -15. Find, in any order, the common difference and the 20th term is -32, find the first term of an arithmetic series is 4. The sum of the first term and the common difference. P 7 Prove that the sum of the sum o first 50 natural numbers is 1275. Problem-solving Use the same method as Example 4. P 8 Show that the sum of the first 2n natural numbers is n(2n + 1). P 9 Prove that the sum of the first n terms is 2225. a Show that $7n^2 + 3n - 4450 =$ 0. b Hence find the value of n. E/P(1 mark) 11 An arithmetic series is given by (k + 1) + (2k + 3) + (3k + 5) + ... + 303 a Find the number of terms in the series in terms of k. 152k + 46208 b Show that the sum of the series is given by k+2 c Given that Sn = 2568, find the value of k. E/P (4 marks) (1 mark) (3 marks) (1 mark) 12 a p (4 marks) c Find, in Calculate the sum of all the multiples of 3 from 3 to 99 inclusive, 3 + 6 + 9 + ... + 99 (3 marks) b In the arithmetic series 4p + 8p + 12p + ... + 400 where p is a positive integer and a factor of 100, i find, in terms of p, an expression for the number of terms in this series. 20 000 ii Show that the sum of this series is 200 + terms of p, the 80th term of the arithmetic sequence (3p + 2), (5p + 3), (7p + 4), ..., giving your answer in its simplest form. (2 marks) 65 Chapter 3 E/P 13 Joanna has some sticks that are all of the same length. She arranges them in shapes as shown opposite and has made the following 3 rows of patterns. Row 1 She notices that 6 sticks are required to make the single pentagon in the first row, 11 sticks in the second row and for the third row she needs 16 sticks. a Find an expression, in terms of n, for the number of sticks required to make a similar arrangement of n pentagons in the nth row. Row 2 Row 3 (3 marks) Joanna continues to make pentagons following the same pattern. She continues until she has completed 10 rows. b Find the total number of sticks Joanna uses in making these 10 rows. (3 marks) Joanna started with 1029 sticks. Given that Joanna continues the pattern to complete the (k + 1)th row: c show that k satisfies (5k - 98)(k + 21) < 0 (4 marks) d find the value of k. (2 marks) Joanna started with 1029 sticks. marks) Challenge An arithmetic sequence has nth term un = $\ln 9 + (n - 1) \ln 3$. Show that the sum of the first n terms = a $\ln 3 n^2 + 3n$ where a is a rational number to be found. 3.3 Geometric sequences a common ratio between Notation A geometric sequences a not terms a common ratio between Notation A geometric sequences a not terms a not terms a common ratio between Notation A geometric sequences a not terms terms. To get from one term to the next you multiply by the common ratio 2, 4, ×2 ×2 1, 6 1, 2 ×1 3 5, 8, ×1 3 16 This is a geometric sequence with common ratio 2. This sequence is decreasing but will never get to zero. ×1 3 -10, 20, -40, 80 ×(-2) ×(-2)

 $\times(-2)$ $\times(-2)$ Here the common ratio is -2. The sequence alternates between positive and negative terms. Notation A geometric sequence with a common ratio |r| < 1 converges. This means it tends to a certain value. You call the value the limit of the sequence with a common ratio |r| < 1 converges. This means it tends to a certain value. is the first term and r is the common ratio. 66 An alternating sequence is a sequence in which terms are alternately positive and negative. Sequences: a 3, 6, 12, 24, ... b 40, -20, 10, -5, ... For this sequence is a sequence a = 3 and r = 63 = 2. a 3, 6, 12, 24, ... i 10th term = 10th terms in the following geometric sequence are 3, 6, 12, 24, ... i 10th terms in the following geometric sequence are 3, 6, 12, 24, ... i 10th term = 63 = 2. a 3, 6 $3 \times 29 = 3 \times 512 = 1536$ iin the term $= 3 \times 2n - 1$ For the 10th term use arn - 1 with a = 3, r = 2 and n = 10. For the nth term use arn - 1 with a = 3, r = 2 and n = 10. For the nth term use arn - 1 with a = 3 and r = -2 b 40, -20, 10, -5, ... 1 i 10th term $= 40 \times (-2)$ 9 Use arn - 1 with a = 40, r = -12 and n = 10. $... 1 = 40 \times -512$ 5 = -12 $(2) n - 1 1 = 5 \times 8 \times (-2) = 5 \times \text{Example} \times (-2) 5 \times \text{Example$ positive, find the exact value of the 11th term in the sequence. nth term is ar, and the 4th term is ar, and the 11th ter the sequence. 8 = 4 ar ar $3 = r^2 = 2$ r = $\sqrt{2}$ You are told in the question that r > 0 so use the positive square root. 67 Chapter 3 Substituting back into equation (1): $a\sqrt{2} = 44$ $a = \sqrt{2}$ $a = 2\sqrt{2}$ Rationalise the denominator. nth term = arn - 1, so 11th term = $(2\sqrt{2})(\sqrt{2})10$ = $64\sqrt{2}$ Simplify your answer as much as possible. Example 10 The numbers 3, x and (x + 6) form the first three terms of a geometric sequence with all positive terms. Find: a the possible values of x, b the 10th term of the sequence with all positive terms. Find: a the possible values of x, b the 10th term of the sequence with all positive terms. Find: a the possible values of x, b the 10th term of the sequence with all positive terms. Find: a the possible values of x, b the 10th term of the sequence with all positive terms. Find: a the possible values of x, b the 10th term of the sequence with all positive terms. a b 10th term = ar9 = $3 \times 29 = 3 \times 512 = 1536$ The 10th term is 1536. Problem-solving In a geometric sequence the ratio between u3 u2 _____ consecutive terms is the same, so ______ u1 = u 2 Simplify the algebraic fraction to form a quadratic equation. \leftarrow Year 1, Section 3.2 Factorise. If there are no negative terms then -3 cannot be an answer. Use the formula nth term = arn-1 with n = 10, x 6 a = 3 and r = = 2.33 Example 11 What is the first term in the general term in term in the general term in the general term in of a geometric sequence. Sequence has a = 3 and r = 2. 68 Sequences and series $3 \times 2n-1 > 1\ 000\ 000\ So\ 2n-1\ \log\ 2n-1 > Divide\ by\ 3.\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ (n-1)\ (n-1)\ \log\ 2 > \log(\ 3\)\ 1\ 000\ 000\ (n-1)\ (n-1)$ log(2) n - 1 > 18.35 (2 d.p.) n > 19.35 n > 20 The 20th term is the first to exceed 1 000 000. Exercise To solve this inequality take logs of both sides. log an = n log a \leftarrow Year 1, Chapter 14 Divide by log 2. n has to be an integer. Online Use your calculator to check your answer. 3C 1 Which of the following are geometric sequences? For the ones that are, give the value of the common ratio, r. a 1, 2, 4, 8, 16, 32, ... b terms of a geometric sequence, find: a the exact value of x, b the exact value of the 4th term. Problem-solving In a geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and nth terms of the following geometric sequence the common u3 u 2 ratio can be calculated by u1 or u 2 4 Find the sixth and u1 a transformed by u1 or u 2 4 Find the sixth and u1 a transformed by u1 or u 2 4 Find the sixth and u1 a transformed by u1 nth term of a geometric sequence is 32 and the first term 4 and third term 1. Find the first term 4 and the common ratio. 69 Chapter 3 7 A geometric sequence is 32 and the 3rd term is 4. Find the first term 4 and third term 1. Find the first term and the common ratio. sequence are given by 8 - x, 2x, and x^2 respectively where x > 0. a Show that $x^3 - 4x^2 = 0$. (2 marks) b Find the value of the sequence is 40. a Show that p satisfies the equation 5 log p + log 5 = 0. (3 marks) b Hence or otherwise, find the value of p correct to 3 significant figures. (1 mark) P 10 A geometric sequence exceeds 500 000. P 11 The first three terms of a geometric sequence are 9, 36, 144. State, with a reason, whether 383 616 is a term in the sequence. Problem-solving Determine the values of a and r and find the general term and solve to find n. If n is an integer, then the number is in the sequence. P 12 The first three terms of a geometric sequence are 3, -12, 48. State, with a reason, whether 49 152 is a term in the sequence. P 13 Find which term in the geometric series is the sum of the terms of a geometric series. The sum of the terms of a geometric series is the sum of the terms of a geometric series. $r \neq 1$ 1-r a(r n - 1) or Sn = $r \neq 1$ r-1 where a is the first term and r is the common ratio. 70 Hint These two formulae are equivalent. It is often easier to use the first one if r < 1 and the second one if r > 1. Sequences and series Example 12 A geometric first n terms of a geometric series is given by the formula a(1 - r n) Sn =of this series is given by $Sn = 1 - rSn = a + ar + ar^2 + ar^3 + ... arn - 2 + ar^{-1} (1)$ Let $rSn = ar + ar^2 + ar^3 + ... arn - 1 + arn (2) (1) - (2)$ gives Sn - rSn = a - arn Sn(1 - r) = a(1 - rn) a(1 - rn) Sn = a - arn Sn(1 - r) = a(1 - rn) a(1 - rn) Sn = a - arn Sn(1 - r) = a(1 - rn) a(1 - rn) Sn = a - arn Sn(1
- r) = a(1 - rn) a(1 - rn) a(1 - rn) Sn = a - arn Sn(1 - r) = a(1 - rn) a(1 - rn) a(1 - rn) Sn = a - arn Sn(1 - r) = a(1 - rn) a(1 series has first term a and common difference r. Prove that the sum of the first n terms a(1 - rn)from Sn. Take out the common factor. Divide by (1 - r). Problem-solving You need to learn this proof for your exam. Example 13 Find the sums of the following geometric series. a 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 - 512 + 256 - 128 + ... + 18 + 54 + ... $= 59\ 048\ 3-1\ b\ Series\ is\ 1024\ -\ 512\ +\ 256\ -\ 128\ +\ ...\ +\ 1\ When\ r > 1\ it\ is\ easier\ to\ use\ the\ formula\ a(rn\ -\ 1)\ Sn\ =\ r-1\ 512\ _\ 1\ So\ a\ =\ 1024\ =\ -\ 2\ and\ the\ nth\ term\ =\ 1\ 1\ 1024\ (-\ 2)\ n-1\ =\ 1\ to\ find\ n.\ (-2)n-1\ =\ 1024\ 2n-1\ =\ 1024\ log\ 1024\ r$ what is given. = 3 and n = 102(310 - 1) So S10 = Sn = 11 - (-2) 11024(1 + 2048) $-1 = \frac{\log 2 n - 1 = 10 n = 11 \text{ So} (-2)n - 1 = (-1)n - 1(2n - 1) = 1024, \text{ so} (-1)n - 1 \text{ must be positive and } 2n - 1 = 1024. 1024 = 210 1 11 1024(1 - (-2)) \frac{\ln 2}{\ln 2} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the formula } a(1 - nn) \text{ Sn} = \frac{11 + \text{When } n < 1, \text{ it is easier to use the$ = 1 + When r < 1, it is easier to use the formula a(1 - rn) Sn =1-r 2 1024.5 =log(2) n > 20.9 It needs 21 terms to exceed 2 000 000. Exercise Problem-solving Determine the values of a and r, then use the formula for the sum of the following geometric series (to 3 d.p. if necessary). a 1 + 2 + 4 + 8 + ... (8 terms) 8 256 2 4 c 3 15 75 234 375 1 e 729 - 243 + 81 - ... - 3 b 32 + 16 + 8 + ... (10 terms) d 4 - 12 + 36 - 108 + ... (6 terms) 5 5 5 5 f - + - ... - 2 4 8 32 768 2 A geometric series has first three terms 3 + 1.2 + 0.48... Evaluate S10. 2 3 A geometric series has first term 5 and common ratio . Find the value of S8, giving your 3 answer to 4 d.p. P 4 The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r. P 5 Find the least value of n such that the sum 3 + 6 + 12 + 24 + ... to n terms exceeds 45. E 3 7 A geometric series has first term 25 and common ratio $_$ 5 Given that the sum to k terms of the series is greater than 61, log (0.024) a show that k > log (0.6) b find the smallest possible value of k. 72 (4 marks) (1 mark) Sequences and series E/P E/P 8 A geometric series has first term a and common ratio r. The sum of the first two terms of the series is 4.48. The sum of the first four terms is 5.1968. Find the two possible values of r. (4 marks) Problem-solving One value will be positive and one value will be negative. 9 The first term of a geometric series is a and the common ratio is $\sqrt{3}$. Show that S10 = 121a($\sqrt{3}$ + 1). (4 marks) E/P 10 A geometric series has first term a and common ratio 2. A geometric series has first term b and common ratio 3. Given that the sum of the first 4 terms of both series 8 is the same, show that a = b. (4 marks) 3 E/P 11 The first three terms of a geometric series are (k - 6), k, (2k + 5), where k is a positive constant. a b c d Show that k2 - 7k - 30 = 0. Hence find the value of k. Find the common ratio of this series. Find the sum of the first 10 terms of this series, giving your answer to the nearest whole number. (4 marks) (2 marks) (1 mark) (2 marks) (2 marks) (1 mark) (2 marks) (3.5 Sum to infinity, the sum of the series is called the sum to infinity. Notation You can write the sum to infinity of a geometric series as S ∞ Consider the sum of the first n terms of the geometric series 1 + 12 + 18 + ... The terms of this series are getting larger, so as n tends to infinity. This is called a divergent series 1 + 12 + 18 + ... The terms of the geometric series 1 + 12 + 18 + ... The terms of this series are getting larger, so as n tends to infinity. this series are getting smaller. As n tends to infinity, Sn gets closer and closer to a finite value, S. This is called a convergent if and only if | r | < 1, condition as -1 < r < 1. where r is the common ratio. a(1 - rn) The sum of the first n terms of a geometric series is given by S n = 1-r a(1-rn) a When |r| < 1, $n \rightarrow \infty$ lim (____) = 1-r 1-r This is because $rn \rightarrow 0$ as $n \rightarrow \infty$. The sum to infinity of a convergent geometric a series is given by $S_{\infty} =$ 1-r Notation lim means 'the limit as n tends to ∞ '. $n \rightarrow \infty$ You can't evaluate the expression when n is ∞ , but as n gets larger the expression gets closer to a fixed (or limiting) value. Watch out You can only use this formula for a convergent series, i.e. when |r| < 1.73 Chapter 3 Example 15 The fourth term is 0.233 28. a Show that this series is convergent. b Find the sum to infinity of the series. a ar3 = 1.08 ar6 = 0.233 28 (1) (2) Dividing (2) by (1): Use the nth term of a geometric series is convergent as |r| = 0.6 < 1.6 Substituting the value of r3 into equation (1) to find a To show that a series is convergent you need to find r, then state that the series is convergent if |r| < 1.0.216a = 1.081.08a = 1.081.0= 151-ra = 161-r16(1-r4) = 15151-r4 = 161r4 = 161r = 6 2(1)(2)a(1-rn)S4 = 15 so use the formula Sn = 16 with n = 4. $1-raS_{\infty} = 16$ so use the formula $S_{\infty} = 16$. 1-rSolve equations simultaneously. a Replace by 16 in equation (1). 1-r1 Take the 4th root of 16 74 Sequences and series 1 b As all terms are positive, r = + 2 a = 16 1 1 - 2 Substitute r = 12 into equation (2) to fnd a. 116(1 - 2) = a = 8 The first term in the series is 8. Exercise 3E 1 For each of the following geometric series: i state, with a reason, whether the series is convergent. ii If the series is convergent. term -5 and sum to infinity -3. Find the common ratio 3. Find the first term, 25 A geometric series has sum to infinity 60 and common ratio -3 and $S^{\infty} = 10$. Find the first term, 25 A geometric series has sum to infinity -3. 7 For a geometric series $a + ar + ar^2 + ...$, S 3 = 9 and S ∞ = 8, find the values of a and r. E/P E/P 8 Given that the geometric series $1 - 2x + 4x^2 - 8x^3 + ...$ is convergent, a find the values of x b find an expression for S ∞ in terms of x. (3 marks) (1 mark) 9 In a convergent geometric series the common ratio is r and the first term is 2. Given that $S \propto = 16 \times S3$, a find the value of the common ratio, giving your answer to 4 significant figures b find the value of the fourth term. (3 marks) (2 marks) 10 The first term of a geometric series is 30. The sum to infinity of the series is 240. 7 a Show that the common ratio,
r, is 8 b Find to 3 significant figures, the difference between the 4th and 5th terms. c Calculate the sum of the first 4 terms, giving your answer to 3 significant figures. The second term of the series is greater than 180. d Calculate the smallest possible value of n. (2 marks) (2 8 and the sum to infinity of the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the two possible values of a. (2 marks) c Find the corresponding two possible values of a. (2 marks) c Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r4 + 15 = 0. (4 marks) b Find the series is 8. 15 a Show that 64r2 - 64r4 + 15 = 0. (4 marks) b Find the series is 8. infinity of a geometric series is 7. A second series is formed by squaring every term in the first geometric series. a Show that the second series is 35, show that the common ratio of the original series is 16 3.6 Sigma notation The Greek capital letter 'sigma' is used to signify a sum. You write it as Σ . You write limits on the top and bottom to show which terms you are summing. This tells you that are summing the expression in brackets with r = 1, r = 2, ... up to r = 5. Look at the limits carefully: they don't have to start at 1. 5 Σ (2r - 3) = -1 + 1 + 3 + 5 + 7 r = 1 7 Σ ($5 \times 2r$) = 40 + 80 + 160 + 320 + 640 r = 3 Substitute r = 1, r = 2, ... up to r = 5. Look at the limits carefully: they don't have to start at 1. 5 Σ (2r - 3) = -1 + 1 + 3 + 5 + 7 r = 1 7 Σ ($5 \times 2r$) = 40 + 80 + 160 + 320 + 640 r = 3 Substitute r = 1, r = 2, ... up to r = 5. Look at the limits carefully: they don't have to start at 1. 5 Σ (2r - 3) = -1 + 1 + 3 + 5 + 7 r = 1 7 Σ ($5 \times 2r$) = 40 + 80 + 160 + 320 + 640 r = 3 Substitute r = 1, r = 2, ... up to r = 5. Look at the limits carefully: they don't have to start at 1. 5 Σ (2r - 3) = -1 + 1 + 3 + 5 + 7 r = 1 7 Σ ($5 \times 2r$) = 40 + 80 + 160 + 320 + 640 r = 3 Substitute r = 1. 1, r = 2, r = 3, r = 4, r = 5 to find the five terms in this arithmetic series. To find the terms in this geometric series, you substitute r = 3, r = 4, r = 5, r = 6, r = 7. You can write some results that you already know using sigma notation: $n \cdot \sum 1 = n$ Hint r = 1 $n \cdot (n + 1) \cdot \sum r = 2$. Calculate $\sum (4r + 1) r = 1.20 \sum (4r + 1) r = 5.4 = 4$ and n = 20.76 Problem-solving Substitute r = 1, 2, etc. to find the terms in the series. Sequences and series $S = (2x + (n - 1)d) 2 20 = (2 \times 5 + (20 - 1)4) 2$ Use the formula for the sum to n terms of an arithmetic series. Substitute n = 5, d = 4 and n = 20 into n = 20 into n = 20 into n = 20. (2a + (n - 1)d). $2 = 10(10 + 19 \times 4) = 10 \times 86$ Online = 860 Check your answer by using your calculator to calculate the sum of the series. Example 18 Find the values of: $12 \ 12 \ b \sum 5 \times 3k - 1 \ a \sum 5 \times 3k - 1 \ b = 1 \ b = 1 \ b = 1 \$ the correct values for a, r and n. k=1 = 5 + 15 + 45 + ... a = 5, r = 3 a(rn - 1) Since r > 1 use the formula Sin = _____ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = _____ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = _____ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = ______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = ______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = ______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = ______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 use the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = _______ and r-1 substitute in a = 5, r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = ______ and r-1 substitute in a = 5. r = 3 and n = 12. 5(3 12 - 1) Since r > 1 are the formula Sin = ______ and r-1 are th 43-1 Problem-solving = 200 When we are summing series from 1 to k - 1. $12 \sum 5 \times 3 k - 1 = 1.328600 - 200 = 1.328400 k = 5$ Exercise $3F_1$ For each series: i write out every term in the series ii hence find the value of the sum. $5a \sum (3r + 1)r = 1.6b \sum 3r$ $2 r = 15 c \sum sin (90r o) r = 181 d \sum 2(-) 3 r = 5 r 2$ For each series: i write the series using sigma notation. 64 4 1 2 a 7 + 13 + 19 + ... + 157 b + 157 3 15 75 46875 4 Evaluate: 20 10 b 53×4 a 5(7 - 2r) r = 1 P c 8 - 1 - 10 - 19 ... - 127 d 57(-) 3r = 1 100 r = 1 r = 1 5 Evaluate: 30 1 a 5(5r - 2) 2 r = 9 200 b 5(3r + 4) r = 100 n P 6 Show that 52r = n + P 7 Show that 52r - 5(2r - 1) = n. r=1 n n r = 1 r = 1 1 ∞ c 5(2r - 8) r 100 Problem-solving 100 c 53×0.5 r d 51 r = 5 r n n k-1 r=k r=1 r=1 Σ ur = Σ ur – Σ ur i=5 n2. 8 Find in terms of k: k k a Σ 4(-2) r b Σ (100 – 2r) r=1 P E/P r=1 k c Σ (7 – 2r) r = 10 ∞ 1 Σ 7(– _) is the sum to infinity of 3 r=1 7 7 7 the geometric series – _ + _ - _ + ... 3 9 27 Hint 9 Find the value of Σ 200 × (_4) ∞ 1 r r = 20 k 10 Given that Σ (8 + 3r) = 377, r r=1 a show that (3k + 58)(k – 13) = 0 b hence find the value of k. E/P (3 marks) (1 mark) k 11 Given that Σ 2 × 3 r = 59 046, r=1 log 19 683 a show that k = ______ log 3 (4 marks) 13 b For this value of k, calculate Σ 2 × 3 r. r = k+1 E/P 12 A geometric series is given by 1 + 3x + 9x2 +... The series is convergent. a Write down the range of possible values of x. ∞ (3 marks) (3 marks) Given that Σ (3x) r – 1 = 2 r=1 b calculate the value of x. Challenge 10 14 r=1 r = 11 Given that Σ (a + (r – 1)d) = Σ (a + (r – 1)d) = \Sigma (b + 0 r + 1) for the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can write a recurrence relation of the next in a sequence you can form un + 1 = f(un) defines each term of a sequence as a function of the previous term. For example, the recurrence relation un + 1 = 2u1 + 3 = 2(6) + 3 = 15 Watch out In order to generate a sequence from a recurrence relation like this, you need to know the first term of the sequence. Example 19 Find the first four terms of the following sequences. a un + 1 = un + 4, u1 = 7 b un + 1 = un + 4, u1 = 7 b un + 1 = un + 4, u1 = 7 b un + 1 = un + 4, u1 = 7 b un + 1 = un + 4, u1 = 5 a un + 1 = un + 4, u1 = 7 b un + 1 = un + 4, u1 = 5 a un + 1 = un + 4, u1 = 5 a un + 1 = un + 4, u1 = 5 a un + 1 = un + 4, u1 = 5 a un + 1 = un + 4,
u1 = 5 a un + 1 = un + 4, un + 1 = un +Substituting n = 1, $u^2 = u^1 + 4 = 5 + 4 = 9$. Substituting n = 2, $u^3 = u^2 + 4 = 9 + 4 = 13$. Substituting n = 3, $u^4 = u^3 + 4 = 13$. Substituting n = 1, 2 and 3. Use u^1 to find u^2 , and then u^2 to find u^3 . This is the same recurrence formula. It produces a different sequence because u^1 is different. Example 20 A sequence a1, a2, a3, ... is defined by a1 = p an + 1 = (an)2 - 1, n > 1 where p < 0. a Show that a3 = p4 - 2p2. 200 c Find \sum ar r=1 a a1 = p a2 = (a1)2 - 1 = p4 - 2p2 b Given that a2 = 0, find the value of p. d Write down the value of a199 Use a2 = (a1)2 - 1 = p4 - 2p2 b Given that a3 = substitute the expression for a 2 to find a 3. 79 Chapter 3 Set the expression for a 2 equal to zero and solve. b $p^2 - 1 = 0$ $p^2 = 1$ $p = \pm 1$ but since p < 0 is given, p = -1 Since this is a recurrence relation, we can see that the sequence is going to alternate between -1 and 0. The first 200 terms will have one hundred -1 and one hundred 0 s. c $a^2 = -1$ a2 = 0, a3 = -1 series alternates between -1 and 0 In 200 terms, there will be one hundred 0s. Problem-solving 200 For an alternating series, consider the sums of the odd and even terms separately. Write the first few terms of the series. The odd terms are 0. Only the odd terms contribute to the sum \sum ar = -100 r=1 d a199 = -1 as 199 is odd Exercise 3G 1 Find the first four terms of the following recurrence relationships. a un + 1 = un + 3, u1 = 1 b un + 1 = un + 3, u1 = 1 b un + 1 = un + 3, u1 = 2 f un + 1 = 2 un + 1, sequences. (Remember to state the first term.) a 3, 5, 7, 9, ... b 20, 17, 14, 11, ... c 1, 2, 4, 8, ... d 100, 25, 6.25, 1.5625, ... e 1, -1, 1, -1, 1, ... f 3, 7, 15, 31, ... g 0, 1, 2, 5, 26, ... h 26, 14, 8, 5, 3.5, ... 3 By writing down the first four terms or otherwise, find the recurrence formula that defines the following sequences: P a un = 2n - 1 b un = 3n + 2 c un = n + 2 n + 1 d un = 2 e un = n2 f un = 3n - 1 4 A sequence of terms is defined for n > 1 by the recurrence relation un + 1 = kun + 2, where k is a constant. Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u1 = 3, a find an expression for u3 Given that u3 = 42: c find the possible values of k. E/P 5 A sequence is defined for n > 1 by the recurrence relation un + 1 = pun + q, u1 = 2 Given that u2 = -1 and u3 = 11, find the values of p and q. 80 (4 marks) Sequences and series E/P 6 A sequence is given by x1 = 2 xn + 1 = xn(p - 3xn) where p is an integer. a Show that x3 = -10p2 + 132p - 432. E/P (2 marks) b Given that x3 = -288 find the value of p. (1 mark) c Hence find the value of x4. (1 mark) 7 A sequence a1, a2, a3, ... is defined by a1 = k an + 1 = 4an + 5 a Find a3 in terms of k. (2 marks) 4 b Show that Σ ar is a multiple of 5. (3 marks) r=1 A sequence is decreasing if un + 1 > un for all $n \in \mathbb{N}$. sequence there is an integer k such that un + k = un for all $n \in \mathbb{N}$. The value k is called the order of the sequence. $\bullet -3$, -6, -12, -24... is an increasing sequence. $\bullet -3$, -6, -12, -24... is a periodic sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing, decreasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing, decreasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of the sequence with a period of 2. $\bullet 1$, -2, 3, -4, 5, -6... is not increasing or periodic. Notation The order of 2. of a periodic sequence is sometimes called its periodic, write down its order. a un + 1 = un + 3, u1 = 7 b un + 1 = un + 3, u1 = 7 b un + 1 = un + 3, u1 = 7 b un + 1 = (un)2, u1 = $_2$ a 7, 10, 13, 16, ... un + 1 > un for all n, so the sequence is increasing. 1 c un = sin(90n°) Write out the first few terms of the sequence. State the condition for an increasing sequence. You could also write that k + 3 > k for all numbers k. 81 Chapter 3 1 1 1 1 b __, __, ... 2 4 16 256 un + 1 < un for all n, so the sequence is decreasing. c u1 = sin(90°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(270°) = -1 u4 = sin(360°) = 0 u5 = sin(450°) = 1 u2 = sin(180°) = 0 u3 = sin(180°) $u = sin(540^\circ) = 0$ $u7 = sin(630^\circ) = -1$ The sequence is periodic, with order 4. Exercise The starting value in the sequence is 0, the period is not 2 because the odd terms alternate between 1 and -1. The graph of y = sin x repeats with period 360°. So sin (x + 360°) = sin x. \leftarrow Year 1, Chapter 9 3H 1 For each sequence is periodic, write down its order. 1 1 1 a 2, 5, 8, 11, 14 b 3, 1, __, __ c 5, 9,
15, 23, 33 3 9 27 d 3, -3, 3, and a sequence is periodic. -3, $3 \ 2$ For each sequence: i write down the first 5 terms of the sequence is increasing, or periodic. iii If the sequence is increasing, or periodic. iii If the sequence is increasing, or periodic. iii If the sequence is norted at 1 = 3 un, 1 = 20 f un + 1 = 3 un, 1 = 20 f un + 1 = 5 - un, 1 = 20 f un + 1 = 3 un, 1 = 20 f un + 1 = 3 un, 1 = 20 f un + 1 = 5 - un, 1 = 20 f un + 1 = 3 un, 1 = 20 f un + 1 = 5 - un, 1 = 20 f un + 1 = 3 un, 1 = 20 f un + 1 = 5 - un, 1 = 20 f un + 1 = 3 un + 3 un sequence of numbers u1, u2, u3, ... is given by un + 1 = kun, u1 = 5. Find the range of values of k for which the sequence is strictly decreasing. E/P 4 The sequence is strictly decreasing. E/P 4 The sequence has nth term an = cos $(90n^\circ)$, n > 1. a Find the order of the sequence is periodic for all positive a and b. Hint Each term in this units and b. Hint Each term in the units and term in the units and b. Hi sequence is defined in terms of the previous two terms. b State the order of the sequence. 3.8 Modelling with series You can model real-life situations with series ach year would form a geometric sequence and the amount they had been paid in total over n years would be modelled by the corresponding geometric series. Example 22 Bruce starts a new company. In year 1 his profits in year 2 are modelled to be £25 000, in year 3, £30 000 and so on. He predicts this will continue until he reaches annual profits of £100 000. He then models his annual profits to remain at £100 000. a Calculate the profits for Bruce's business in the first 20 years. b State one reason why this may not be a suitable model. c Bruce's business in the first 20 years. a Year 1 P = 20 000, Year 2 P = 25 000, Year 3 P = 30 000 a = 20 000, d = 5000 un = a + (n - 1)d 100 000 = 20 000 + (n - 1)(5000) 100 000 = 20 000 + (n reach £100 000. 85 000 = 5000n 85 000 n == 17500017S17 = (2(20000) + (17 - 1)(5000)) 2 = 1020000 Solve to find n. You want to know how much he made overall in the 17 years, so find the sum of the arithmetic series. S20 = 1020000 + 3(100000) = 1320000 So Bruce's total profit after 20 years is £1320000. In the 18th, 19th formula for the sum of the first n terms a(rn - 1) of a geometric series Sn = r - 1 20000(1.0520 - 1) S20 = r - 1 2000(1.0520 - 1) S200(1.0500 - 1) S200(1.05001.05 - 1 S20 = 661 319.08 So Bruce's total profit after 20 years is £661 319.08. Example 23 A piece of A4 paper is folded in half repeatedly. The thickness of the A4 paper is 0.5 mm. a Work out the thickness of the paper after four folds. b Work out the thickness of the paper after 20 folds. c State one reason why this might be an unrealistic model. a a = 0.5 mm, r = 2 After 4 folds: $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence, as each time we fold the paper the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text{ mm}$ This is a geometric sequence the thickness doubles. b After 20 folds $u_2 = 0.5 \times 24 = 8 \text$ (after 0 folds), u2 is after 1 fold, so u5 is after 4 folds. c It is impossible to fold the paper that many times so the model is unrealistic. Exercise Problem-solving If you have to comment on the validity of a model, always refer to the context given in the question. 3I 1 An investor puts £4000 in an account. Every month thereafter she deposits another £200. How much money in total will she have invested at the start of a the 10th month and b the mth month? P 2 Carol starts a new job on a salary of £20 000. She is given an annual wage rise of £500 at the end of every year until she reaches her maximum salary of £25 000. Find the total amount she earns (assuming no other rises), a in the first 10 years, b over 15 years and c state one reason why this may be an unsuitable model. 84 Hint At the start of the 6th month she will have only made 5 deposits of £200. First find how many years it will take her to reach her maximum salary. Sequences and series P 3 James decides to save some money during the six-week holiday. He saves 1p on the first day, 2p on the second, 3p on the third and so on. a How much will he have at the end of the holiday (42 days)? b If he carried on, how long would it be before he has saved £100? P 4 A population of ants is growing at a rate of 10% a year. If there were 200 ants in the initial population, write down the number of ants after: a 1 years c 3 years c 3 years c 3 years. This is a geometric sequence. a = 200 and r = 1.1 P 5 A motorcycle has four gear is 120 km h-1. Given that the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in bottom gear is 40 km
h-1 and the maximum speed in bottom gear is 40 km h-1 and the maximum speed in b form a geometric progression, calculate, in km h-1 to one decimal place, the maximum speeds in the two intermediate gears. P 6 A car depreciates in value first be less than £5000? E Problem-solving Use your answer to part a to write an inequality, then solve it using logarithms. 7 A salesman is paid commission of £10 per week for each life insurance policy that he has sold. Each week he sells one new policy so that he is paid £10 commission in the first week, £20 commission in the second week, £30 commission in the third week and so on. a Find his total commission in the first year of 52 weeks. (2 marks) b In the second year the commission increases to £11 per week on new policies sold, although it remains at £10 per week. Show that he is paid £542 in the second year. (3 marks) c Find the total commission paid to him in the second year. E (2 marks) 8 Prospectors are drilling to a depth of 50 m is £1140. Each subsequent extra depth of 50 m costs £40 and, hence, the total cost of drilling to a depth of 50 m is £1140. 300. (3 marks) b The total sum of money available for drilling is £76 000. Find, to the nearest 50 m, the greatest depth that can be drilled. (3 marks) E 9 Each year, for 40 years, Anne will pay money into a savings scheme. In the first year she pays in £500. Her payments then increase by £50 each year, so that she pays in £550 in the second year, £600 years, £600 in the third year, and so on. a Find the amount that Anne will pay in the 40th years. (2 marks) b Find the total amount that Anne will pay in over the savings scheme. In the first year he pays in £890 and his payments then increase by £d each year. Given that Brian and Anne will pay in exactly the same amount over the 40 years, find the value of d. (4 marks) 85 Chapter 3 P 10 A virus is spreading such that the number of people infected increases by 4% a day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected? P 11 I invest £A in the bank at a rate of interest of 3.5% per annum. How long will it be before I double my money? P 12 The fish in a particular area of the North Sea are being reduced by 6% each year due to overfishing. How long will it be before the fish stocks are halved? P 13 The man who invented the game of chess was asked to name his reward. He asked for 1 grain of corn to be placed on the first square of his chessboard, 2 on the second, 4 on the third and so on until all 64 squares were covered. He then said he would like as many grains of corn did he claim as his prize? P 14 A ball is dropped from a height of 10 m. It bounces to a height of 7 m and continues to bounce. Subsequent heights to which it bounces follow a geometric sequence. Find out: a how high it will bounce b the total vertical distance travelled up to the point when the ball hits the ground for the sixth time. P 15 Richard is doing a sponsored cycle. He plans to cycle 1000 miles over a number of days. He plans to cycle 10 miles on day 1 and increase the distance by 10% a day. a How long will it take Richard to complete the challenge? b What will be his greatest number of miles completed in a day? P 16 A savings scheme is offering a rate of interest of 3.5% per annum for the lifetime of the plan. Alan wants to save up £20 000. He works out that he can afford to save £500 every year. which he will deposit on 1 January. If interest is paid on 31 December, how many years will it be before he has saved up his £20 000? Mixed exercise E/P E/P 3 1 A geometric series has third term 8. 2 a Show that the common ratio of the series is _3 b Find the first term of the series. c Find the series. d Find the series. d Find the difference between the sum of the first 10 terms of the series and the series and the series is 80 and the fifth term of the series is 5.12. a Show that the common ratio of the series is 0.4. Calculate: b the first term of the series 86 (2 marks) marks) (2 marks) Sequences and series c the sum to infinity of the series, giving your answer as an exact fraction d the difference between the sum to infinity of the series, giving your answer in the form a × 10n, where 1 < a , 10 and n is an integer. E/P 3 The nth term of a sequence is un, where un = 95(5), $n = 1, 2, 3, \dots$ a Find the values of u1 and u2. Giving your answers to 3 significant figures, calculate: b the value of u21 (2 marks) n = 1 d the sum to infinity of the series whose first term is u1 and whose nth term is un. 2 n b Show that $\sum un = -9.014$ to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (2 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (3 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (3 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (3 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (3 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (3 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 significant figures. (3 marks) 2un - 1 c Prove that un + 1 = -9.014 to 4 si of the series, b the first term of the series, c the sum to infinity of the series. d Calculate the difference between the sum to infinity of the series. B Find when the value will be less than £4000. 7 The first three terms of a geometric series are p(3q + 1), p(2q + 2) and p(2q - 1), where p and q are non-zero constants. a Show that one possible value. b Given that q = 5, and the sum to infinity of the series is 896, find the sum of the first 12 terms of the series. Give your answer to 2 decimal places. 8 a Prove that the sum of the first n terms in an arithmetic series is n S = (2a + (n - 1)d) 2 where a = first term and d = common difference. b Use this to find the sum of the first 100 natural numbers. n 9 Find the least value of n for which $\sum (4r - 3) > 2000$. r=1 (2 marks) (2 mar marks) (3 marks) (2 marks) (2 marks) (2 marks) 87 Chapter 3 E/P 10 The sum of the first two terms of an arithmetic series is 47. The thirtieth term of the series is 47. The thirtieth term of the series and the common difference b the sum of the first 60 terms of the series. lie between 1 and 400. (3 marks) b Hence, or otherwise, find the sum of the integers, from 1 to 400 inclusive, which are not divisible by 3. (2 marks) E/P 12 A polygon has 10 sides. The lengths of the sides, starting with the shortest, form an arithmetic series. The perimeter of the polygon is 675 cm and the length of the longest side is twice that of the shortest side. Find the length of the shortest side of the polygon. (4 marks) E/P 13 Prove that the sum of the first 2n multiples of 4 is 4n(2n + 1). E/P 14 A sequence of numbers is defined, for n > 1, by the recurrence relation un+1 = kun - 4, where k is a constant. Given that u1 = 2: (2 marks) a find expressions, in terms of k, for u2 and u3. (2 marks) below that u1 = 2: (2 marks) a find expressions, in terms of k, for u2 and u3. (2 marks) below that u1 = 2: (2 marks) a find expressions, in terms of k, for u2 and u3. (2 marks) below that u1 = 2: (2 marks) a find expressions, in terms of k, for u2 and u3. (2 marks) below that u1 = 2: (2 Given also that u3 = 26, use algebra to find the possible values of k. E/P 15 The fifth term of the series is -6 and calculate the common difference of the series. (3 marks) b Given that the nth term of the series is greater than 282, find the least possible value of n. (3 marks) E/P 16 The fourth term of an arithmetic series is 3k, where k is a constant, and the sum of the series is 9 - 8k. (3 marks) b Find an expression for the common difference of the series in terms of k. (2 marks) Given
that the seventh term of the series is 12, calculate: c the value of k (2 marks) d the sum of the first 20 terms of the series. (2 marks) E/P 17 A sequence is defined by the recurrence relation 1 an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an , a1 = p a b E/P Show that the sequence is defined by a1 = k an + 1 = __ an + 2an + 6, n > 1 where k is an integer. 88 (4 marks) (2 marks) (2 marks) (2 marks) R = 1 E/P 19 The first term of a geometric series is 130. The sum to infinity of the series is 650. a Show that the common ratio, r, is 5 4 (3 marks) b Find, to 2 decimal places, the difference between the 7th and 8th terms. (2 marks) r terms. (2 marks) The sum of the first n terms of the series is greater than 600. -log 13 d Show that n > _____ adult population of a town is 25 000 at the beginning of 2012. A model predicts that the adult population of the town will increase by 2% each year, forming a geometric sequence. a Show that the predicted population at the beginning of 2014 is 26 010. (1 mark) The model predicts that after n years, the population will first exceed 50 000. log 2 (3 log 1.02 c Find the year in which the population first exceeds 50 000. (2 marks) d Every member of the adult population is modelled to visit the doctor has from the beginning of 2012 to the end of 2019. (4 marks) e Give a reason why this model for doctors appointments may not be appropriate. (1 mark) E/P 21 Kyle is making some patterns out of squares. He has made 3 rows so far. a Find an expression, in terms of n, for the number of squares required to make a similar arrangement in the nth row. (3 marks) b Kyle counts the number of squares used to make the pattern in the kth row. He counts 301 squares. Write down the value of k. (1 mark) c In the first q rows, Kyle uses a total of p squares. (3 marks) i Show that q2 + 2q - p = 0. ii Given that p > 1520, find the minimum number of rows that Kyle makes. (3 marks) E/P Row 1 Row 2 Row 3 22 A convergent geometric series has first term a and common ratio r. The second term of the series is -3 and the sum to infinity of the series is 6.75. (4 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) c Find the sum of the first 5 terms of the series, giving your answer to 2 decimal places. (3 marks) 89 Chapter 3 Challenge A sequence is defined by the recurrence relation u n + 2 = 5un + 1 - 12 = 0. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that the series is convergent, find the value of r. (2 marks) a Show that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given that $27r^2 - 27r - 12 = 0$. b Given 6un. a Prove that any sequence of the form un = p × 3n + q × 2n, where p and q are constants, satisfies this recurrence relation. Given that u1 = 5 and u 2 = 12, b find an expression for un in terms of n only. c Hence determine the number of digits in u100. Summary of key points 1 In an arithmetic sequence, the difference between consecutive terms is constant. 2 The formula for the nth term of an arithmetic sequence is un = a + (n - 1)d, where a is the first term and d is the common difference. 3 An arithmetic series is given by Sn = (2a + (n - 1)d), where a is the first term and d is the first term. and d is the common difference. n You can also write this formula as Sn = (a + 1), where l is the last term. 2 4 A geometric sequence is un = arn - 1, where a is the first term and r is the common ratio. 6 The sum of the first n terms of a geometric series is given by a(1 - rn) a(rn - 1) or Sn =, $r \neq 1$ 1 - r - 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$, $r \neq 1$ 1 - r - 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$, $r \neq 1$ Sn =, $r \neq 1$ 1 - r - 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$, $r \neq 1$ 1 - r - 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the common ratio. 7 A geometric series is given by $S^{\infty} =$. 1 - r = 1 where a is the first term and r is the com $n \in \mathbb{N}$. A sequence is periodic if the terms repeat in a cycle. For a periodic sequence there is an integer k such that un + k = un for all $n \in \mathbb{N}$. The value k is called the order of the sequence. 90 4 Binomial expansion Objectives After completing this chapter you should be able to: • Expand (1 + x)n for any rational constant n and determine the range pages 92-97 of values of x for which the expansion is valid • Expand (a + bx)n for any rational constant n and determine the range - pages 97-100 of values of x for which the expansion is valid • Use partial fractions to expand fractional expressions - pages 101-103 Prior knowledge check 1 Expand the following expressions in ascending powers of x up to and including the term in x3. a (1 + 5x)7 b (5 - 2x)10 c $(1 - x)(2 + x)6 \leftarrow$ Year 1, Chapter 8 The binomial expansion can be used to find polynomial approximations for expressions involving fractional and negative indices. Medical physicists use these approximations to analyse magnetic fields in an MRI scanner. 2 Write each of the following $(1 + 2x)(1 - 5x) 24x^2 + 48x + 24c$ $(1 + x)(4 - 3x)^2 24x - 1b$ $2(1 + 2x) \leftarrow$ Sections 1.3 and 1.4 91 Chapter 4 4.1 Expanding (1 + x)n If n is a natural number you can find the binomial expansion for (a + bx)n using the formula: n n n $(a + b)n = an + () + an - 1b + () + an - 2b^2 + an$ using partial fractions. -14x + 7a $\dots +
() + an - rbr + \dots + bn$, $(n \in \mathbb{N}) 2 1 r$ If n is a fraction or a negative number you need to use this version of the binomial expansion. Hint There are n + 1 terms, so this formula produces a finite number of terms. This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series. n(n - 1) n(n - 1)(n $x^2 + \dots + ()xr + \dots + ()xr + \dots, (|x| < 1, n \in \mathbb{R})$ r 2! 3! The expansion is valid when |x| < 1. When n is not a natural number, none of the factors in the expression $n(n - 1) \dots (n - r + 1)$ are equal to zero. This means that this version of the binomial expansion produces an infinite number of terms. Example Watch out This expansion is valid for any real value of n, but is only valid for values of x that satisfy |x| < 1, or in other words, when -1 < x < 1. 1 1 Find the first four terms in the binomial expansion of 1+x 1-1x3 + ... = 1 - x + x2 - x3 + ... Replace n by -1 in the expansion. As n is not a positive integer, no coefficient will ever be equal to zero. Therefore, the expansion of (1 + bx)n, where n is negative or a fraction, is valid for |bx| < 1, or |x| < -b 92 Binomial expansion Example 2 condition |x| < 12 Substitute $\overline{x} = 0.01$ into both sides of the expansion. Simplify both sides. Note that the terms are getting smaller. 98 Write 0.98 as _____100 Use rules of surds. This approximation is accurate to 7 decimal places. Binomial expansion Example 42+x ______f(x) = _____f(x) = ____f(x) = ___f(x) = __f(x) = __f(x x^2 or $x \times x$. Add these together to 2.8 find the term in x2.5 In the expansion of (1 + kx) - 4 the coefficient of x is 20. a Find the value of k. b Find the value of k. b Find the corresponding coefficient of the x2 term. $(-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + (-4)(-5) = (1 + kx) - 4 = 1 + (-4)(kx) + \dots + ($ Solve to find k. b Coefficient of $x^2 = 10(-5)^2 = 250\ 95\ Chapter 4\ Exercise 4A\ 1$ For each of the following, i find the binomial expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to and including the x3 term ii state the range of values of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to an a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state the range of x for which the expansion up to a state find the binomial expansion up to and including the x3 term ii state the range of values of x for which the expansion is valid. a (1 + 3x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) 4 f (1 - 4x) - 3 d (1 - 5x) - 3 b (1 + 2x) - 3 d (1 - 5x) - 3 b (1 - 5xBinomial expansion P E/P $\sqrt{$ 1+x 7 Show that if x is small, the expression 1-x 1 is approximated by 1 + x + 2 x2. Notation 'x is small' means we can assume the expansion is valid for the x values being considered as, high powers become insignificant compared to the first few terms. 6 4 8 h(x) = ____ 1 + 5x 1 - 3x a Find the series expansion of h(x), in ascending powers of x, up to and including the x2 term. Simplify each term. b Find the percentage error made in using the series expansion in part a to estimate the value of h(0.01). Give your answer to 2 significant figures. (6 marks) (3 marks) c Explain why it is not valid to use the expansion to find h(0.5). E/P (1 mark) _3 9 approximate value of $\sqrt{10}$. Write your answer to 2 decimal places. 4.2 Expanding (a + bx)n for any constants a and b. You need to take a factor of an out of the expression: b b = (a(1 + x)) = an(1 + x)) = an(1 + x) n (a + bx)n for any constants a and b. You need to take a factor of an out of the expression: b b = (a(1 + x)) = an(1 + x)) = an(1 + x) n (a + bx)n for any constants a and b. You need to take a factor of an out of the expression: b b = (a(1 + x)) = an(1 + x) n (a + bx)n for any constants a and b. You need to take a factor of an out of the expression of (1 + x) n (a + bx) n for any constants a and b. You need to take a factor of an out of the expression of (1 + x) n (a + bx) n for any constants a and b. You need to take a factor of an out of the expression of (1 + x) n (a + bx) n (a + bx) n (a + bx) n (b + the expansion of (1 + x). a 97 Chapter 4 |a | b a The expansion of (a + bx)n, where n is negative or a fraction, is valid for x < 1 or |x| < < $\frac{1}{2} = \frac{1}{2} + \frac{1$ values of x for which the expansion is valid. $11 a \sqrt{4} + 2x b$ c 22+x (4-x) Hint Write part g 51 + x 11 a s 1 - f e g $x + 23 + 2x 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $h 2 + x \sqrt{2} + x$ $d \sqrt{9} + x$ $d \sqrt{9$ and including the (5 marks) term in x3. Give each coefficient as a simplified fraction. $3 m(x) = \sqrt{4 - x}$, |x| < 4 a Find the series expansion of m(x), in ascending powers of x, up to and including the x2 term. Simplify each term. $\sqrt{351}$ b Show that, when x =, the exact value of m(x) is 3 9 (4 marks) (2 marks) c Use your answer to part a to find an approximate value for $\sqrt{35}$, and calculate the percentage error in your approximation. (4 marks) P 1 1 1 are $3 + x + x^2 + \dots 4$ The first three terms in the binomial expansion of $3 18 \sqrt{a + bx}$ a Find the values of the constants a and b. b Find the coefficient of the x3 term in the expansion. P $3 + 2x - x2 5 f(x) = 10^{-10} f(x)$ 4-x 5 3 11 Prove that if x is sufficiently small, f(x) may be approximated by + x - x2.416 64 E/P E/P 51 - x, where |x| < -, in ascending powers of x up to and including the term in x2, 6 a Expand $2\sqrt{5} + 2x$ giving each coefficient in simplified surd form. (5 marks) 2x - 1 - x2. terms in the expansion of as a series in ascending $\sqrt{5} + 2x$ powers of x. (4 marks 16 1 7 a Use the binomial theorem to expand (16 - 3x) 4, |x| < x in ascending powers of x, 3 up to and including the term in x2, giving each term as a simplified fraction. 4 approximation to $\sqrt{15.7}$. Give your answer to 3 decimal places. (2 marks) 3 2 1 8 g(x) = ______, |x| < __4 - 2x 3 + 5x 2 a Show that the first three terms in the series expansion of g(x) can be written 719 1 107 as ______ + ___x - ____x 2.432 12 72 b Find the exact value of g(0.01). Round your answer to 7 decimal places. c Find the percentage error made in using the series expansion in part a to estimate the value of g(0.01). Give your answer to 2 significant figures. 100 (5 marks) (2 marks) (2 marks) (2 marks) (2 marks) (3 marks) Binomial expansion 4.3 Using partial fractions can be used to simplify the expansions of more difficult expressions. Links You need to be confident expressing algebraic fractions as sums of partial fractions. \leftarrow Chapter 1 Example 7 4 - 5x a Express A=34 - 5x32will help you stay organised and check your answers. Expand 3(1 + x) - 1 using the binomial expansion with n = -1. = $3(1 - x + x^2 - x^3 + ...) = 3 - 3x + 3x^2 - 3x^3 + ... 101$ Chapter 4 The expansion of 2(2 - x) - 1x - 1 = 2(2(1 - x)) + 2(1 - x) + 2(1 - x2 2! x 3 + (-1)(-2)(-3)(-)+ ...) 2 Take out a factor of 2-1. x - 1 Expand (1 -) using the binomial expansion 2 x with n = -1 and $x = 2x x^2 x^3 = 1 \times (1 + + + + ...) 2 4 8 x x^2 x^3 = 1 + + 2 4 8 4 - 5x$ Hence (1 + x)(2 - x) = 3(1 + x) - 1 - 2(2 - x)
- 1 'Add' both expressions. = $(3 - 3x + 3x^2 - 3x^3) \times x^2 \times x^3 - (1 + x^2 + x^2) + 24872511$ $= 2 - x + x^2 - x^3 2 4 8 3 c$ is valid if $|x| < 11 + x^2 - x^3 z 4 8 3 c$ is valid if $|x| < 11 + x^2 - 2 - 1012$ The expansion is infinite, and converges when |x| < 2.2 || Watch out You need to find the range of values of x that satisfy both inequalities. The expansion is valid as partial fractions. (1 - x)(2 + x) 8x + 4 b Hence or otherwise expand in ascending powers of x as far as the term in x^2 . (1 - x)(2 + x) c State the set of values of x for which the expansion is valid. 102 Binomial expansion P 2x 2 a Express – 2 as partial fractions. (2 + x) 2xwhen |x| < 1. Exercise P 4C 8x + 4 1 a Express 1 b Hence prove that - 2 can be expressed in the form - x + Bx2 + Cx3 where constants B 2 (2 + x) and C are to be determined. c State the set of values of x for which the expansion is valid. P E/P P 6 + 7x + 5x2 3 a Express ______as partial fractions. (1 + x)(1 - x)(2 + x) 6 + 7x + 5x2 b Hence or otherwise expand in ascending powers of x as far as the term in x3. (1 + x)(1 - x)(2 + x) c State the set of values of x for which the expansion is valid. $12x - 1 \ 14 \ g(x) =$ ______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) can be expressed in the form g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < __3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ___3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ___3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ___3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ___3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ___3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x) =______, $|x| < ____3 (1 + 2x)(1 - 3x)$ A B Given that g(x)in ascending powers of x, as far as the term in x2. (x + 3)(x - 2)Give each coefficient as a simplified fraction. (7 marks) E/P 2x 2 + 5x + 11 1 , $|x| < _7 f(x) = _7 f(x)$ 2(2x-1)2(x+1)CBAf(x) can be expressed in the form f(x) = 2 + 2x - 1(2x - 1)x + 1 a Find the values of A, B and C. (4 marks) b Hence or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in x2. Simplify each term. (6 marks) c Find the percentage error made in using the series expansion in part b to estimate the value of f(0.05). Give your answer to 2 significant figures. (4 marks) 103 Chapter 4 Mixed exercise P 4 1 For each of the following, i find the binomial expansion up to and including the x3 term ii state the range of values of x for which the expansion is valid. $1 a (1 - 4x) 3 b \sqrt{16} + x c$ $1 - 2x E 1 + x g \sqrt{16} + x c$ 1 - 2x E 1 + x c 1(1 - x)(1 - 2x) 1 2 Use the binomial expansion to expand (1 - x), |x| < 2 in ascending powers of x, 2 3 up to and including the term in x, simplifying each term. 1 2 (5 marks) 1 3 a Give the binomial expansion of (1 + x) 2 up to and including the term in x3. b By substituting x = 4, find an approximation to $\sqrt{5}$ as a fraction. 1 E/P 2 4 The binomial expansion of (1 + 9x) 3 in ascending powers of x up to and including the term in 1 x3 is 1 + 6x + cx2 + dx3, |x| < 9 a Find the value of c and the value of d. (4 marks) b Use this expansion with your values of c and d together with an appropriate value 2 of x to obtain an estimate of (1.45) 3. (2 marks) 2 c Obtain (1.45) 3 from your calculator and hence make a comment on the accuracy of the estimate you obtained in part b. P (1 mark) 1 5 In the expansion of (1 + ax) 2 the coefficient of x2 is -2. a Find the possible values of a. b Find the corresponding coefficients of the x3 term. E 6 f(x) = (1 + 3x) - 1, $|x| < _3 1$ a Expand f(x) in ascending powers of x up to and including the term in x3. (5 marks) b Hence show that, for small x: 1 + x (4 marks) $\approx 1 - 2x + 6x^2 - 18x^3$. 1 + 3x c Taking a suitable value for x, which should be stated, use the series expansion in 101 part b to find an approximate value for 103, giving your answer to 5 decimal places. (3 marks) E/P 7 When (1 + ax)n is expanded as a series in ascending powers of x, the coefficients of x and x2 are -6 and 27 respectively. a Find the values of a and n. (4 marks) b Find the coefficients of (3 marks) x3. c State the values of x for which the expansion is valid. 104 (1 mark) Binomial expansion 3 3 3 9 ______ can be approximated by ______ x + _____ x 2. 8 Show that if x is sufficiently small then ______ 2 16 256 $\sqrt{4}$ + x E E 1 ______, where |x| < 4, in ascending powers of x up to and including the term in x2. 9 a Expand ______ $\sqrt{4}$ - x Simplify each term. (5 marks) 1 ______ + 2x as a series in b Hence, or otherwise, find the first 3 terms in the expansion of x. (4 marks) 10 a Find the first four terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of 2 (2 + 3x) - 1, $|x| < _3$ (4 marks) b Hence or otherwise, find the first four non-ze (3 marks), |x| < 32 + 3x E/P 1 11 a Use the binomial theorem to expand (4 + x) - 2, |x| < 4, in ascending powers of x, up to and including the x3 term, giving each answer as a simplified fraction. (5 marks) b Use your expansion, together with a suitable value of x, to obtain an approximation $\sqrt{2}$ to . Give your answer to 4 decimal 1 + 4x (1 + 4x) a Find the values of A and B. (3 marks) b Hence, or otherwise, find the series expansion of f(x), in
ascending powers of x, up to and including the term x2, simplifying each term. (6 marks) 105 Chapter 4 E/P 9x2 + 26x + 20 15 q(x) = ______, |x| < 1 (1 + x)(2 + x) a Show that the expansion of q(x) in ascending powers of x can be approximated to 10 - 2x + Bx2 + Cx3 where B and C are constants to be found. (7 marks) b Find the percentage error made in using the series expansion in part b to estimate the value of q(0.1). Give your answer to 2 significant figures. (4 marks) Challenge Obtain the first four non-zero terms in the expansion, in ascending 1 2 powers of x, of the x < 1, $\sqrt{1 + 3x^2}$ Summary of key points 1 This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series: $n(n - 1)x^2 n(1 + x)n = 1 + nx + \dots + (r)x^r +$ function f(x) where f(x) =< 1. 2 1 The expansion of (1 + bx)n, where n is negative or a fraction, is valid for |bx| < 1, or |x| < b 3 b a The expansion of (a + bx)n, where n is negative or a fraction, is valid for a x < 1 or |x| < b 3 b a The expansion of (a + bx)n, where n is negative or a fraction, is valid for |bx| < 1, or |x| < b 3 b a The expansion of (a + bx)n, where n is negative or a fraction, is valid for a x < 1 or |x| < b 3 b a The expansion of (a + bx)n, where n is negative or a fraction, is valid for a x < 1 or |x| < b 3 b a The expansion of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n, where n is negative or a fraction of (a + bx)n and (a +equation $x^2 - 2 = 0$ has no rational solutions. You may assume that if n2 is an even integer then n is also an even integer. (4) \leftarrow Section 1.4 14x2 + 13x + 2 8 2 (x + 1)(2x + 1) Find the values of the constants A, B \leftarrow Section 1.4 and C. \leftarrow Section 1.1 E 4x 1 3 Express as a single 2 2 x - 2x - 3 x + x fraction in its simplest form. (4) \leftarrow Section 1.2 p 3 3 4 f(x) = 1 - _____ + ____ 2, x \neq -2 x + 2 (x + 2) x2 + x + 1, x \neq -2. a Show that f(x) = ______ (x + 2)2 b Show that x2 + x + 1 > 0 for all values of x, x \neq -2. E E \leftarrow Section 1.3 E/p \leftarrow Section 1.3 P - 3x - 12 x 2 10 p(x) = ______ (1 - x)(1 + 2x) C B A + ______ (x + 1)(x + 2)(x + 3) x + 1 x + 2 x + 3 where P, Q and R are constants, find the values of P, Q and R. (4) ex + f 3 x 2 + 6x - 2 Given that ______; d + _____ 2 x + 4 x2 + 4 find the values of d, e and f. (4) Show that p(x) can be written in the form \leftarrow Section 1.5 c Show that f(x) > 0 for all values of x. E 2 f(x) = ______ 2, x \neq -1, x \neq 2. (2 - x) (1 + x) Find the values of A, B and C such that C B A (4) $f(x) = + 22 - x + x (1 + x) \leftarrow$ Section 2.1 12 The function p(x) is defined by 4x + 5, $x_{2} - 2x + 4$, x > -2 a Sketch p(x), stating its range. (3) b Find the exact values of a such that p(a) = -20. (4) \leftarrow Section 2.2 107 Review exercise 1 E/p 13 The functions p and q are defined by 1 p(x) = _____, x \in \mathbb{R}, x \neq -4 x+4 q(x) = 2x - 5, x \in \mathbb{R} E y = f(x) a Find an expression for qp(x) in the ax + b form _____(3) cx + d b Solve qp(x) = 15. (3) Let r(x) = qp(x). c Find r-1(x), stating its domain. -5 14 The functions f and g are defined by: $x+2 f: x \mapsto ____x x \neq 0.5 g: x \mapsto \ln (2x - 5), x \in \mathbb{R}$, $x \neq 0.5 g: x \mapsto \ln (2x - 5), x \mapsto \ln (2x$ x. 2 a Sketch the graph of f. 3x + 2 b Show that $f_2(x) = x+21$ c Find the exact value of gf(4). d Find g-1(x), stating its domain. 5 x The point M (2, 4) is the maximum turning point of the graph. (3) (3) a y = f(x) + 3 b y = |f(x)|(2) c y = f(|x|)(2) (2) Show on each graph the coordinates of any maximum turning points. (2) \leftarrow Sections 2.5, 2.6 (3) E/p 17 The function h is defined by: p(x) = 3x + b, $x \in \mathbb{R}$ a Draw a sketch of y = h(x), labelling the turning points and the x- and y-intercepts. (4) 15 The functions p and g are defined by: p(x) = 3x + b, $x \in \mathbb{R}$ Given that p(x) = 1 - 2x, $x \in \mathbb{R}$ Given that p(x) = 1 - 2x, $x \in \mathbb{R}$ a Draw a sketch of y = h(x), and q - 1(x) (3) 2 O Sketch, on separate diagrams, the graphs of: \leftarrow Section 2.3, 2.4 E/p M (2, 4) The figure shows the graph of y = f(x), -5 < x < 5 (3) \leftarrow Section 2.3 E/p y 16 c show that ax + b p - 1q - 1(x) = q - 1p - 1(x) = d + 1p - 1(x) = q - 1p - 1(x) = 1 + 1p - 1(x) = 1 +f(x) = -3 |x + 4| + 8, $x \in \mathbb{R}$ 5 The diagram shows a sketch of the graph y = f(x). The curve has a minimum at the point A (1, -1), passes through x-axis at the origin, and the point B (2, 0) and C (5, 0); the asymptotes have equations x = 3 and y = 2. y = f(x) (1) a State the range of f. b Give a reason why f (x) does not exist. (1) -1 a Sketch, on separate axes, the graphs of: i y = |f(x)|(2) ii y = -f(x + 1)(2) iii y = -f(x + 1)(2)(x + 1)solutions to each equation. E/p i 3|f(x)| = 2 (2) ii 2|f(x)| = 3. (2) \leftarrow Section 2.7 21 The 4th, 5th and 6th terms in an arithmetic sequence are: 12 - 7k, 3k 2 - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k 2 - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k 2 - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k 2 - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k 2 - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k 2 - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k, 3k - 10k a Find two possible values of k. (3) Given that the sequence are: 12 - 7k a Given that the sequenc b find the first term and
the common difference. (2) g(x) = 2 |x + b| - 3, b, 01 \leftarrow Section 3.1 v E v = g(x) (0, 32) O x B a Show that $3n_2 - 165n + 2250 = 0$. A The graph cuts the v-axis at (0, 2), b Find the two possible values for n, 3 a Find the value of b, 22 The 4th term of an arithmetic sequence is 72. The 11th term is 51. The sum of the first n terms is 1125. (2) (4) (2) \leftarrow Section 3.2 109 Review exercise 1 E/p 23 a Find, in terms of p, the 30th term of the arithmetic sequence a Find the two possible values of x and the corresponding values of the common ratio. (4) (19p - 18), (17p - 8), (17 b Given S31 = 0, find the value of p. (1) b find the first term (3) c find the sum to infinity of the series. (2) \leftarrow Sections 3.1, 3.2 \leftarrow Sections 3.3, 3.5 E/p 24 The second term of a geometric sequence is 256. The eighth term of the same sequ value of r correct to 3 significant figures. $E/p(3)(3) \leftarrow$ Section 3.3 28 A sequence a1, a2, a3, ... is defined by a1 = k, an + 1 = 3an + 5, n > 1 where k is a positive integer. a Write down an expression for a2 in terms of k. (1) b Show that 3 = 9k + 20.4 c i Find 3terms of a geometric sequence are 10, 50 6 and 250 36 r=1. E/p a Find the sum to infinity of the series. (3) Given that the sum to k terms of the series is greater than 55, 1 log() 12 b show that k. 5 log() 6 (4) c find the smallest possible value of k. (1) \leftarrow Sections 3.4, 3.5 E/p a Show that 4r2 + 4r - 3 = 0. (3) b Find the two possible values of r. (2) Given that r is positive, c find the sum to infinity of the series. (2)

Sections 3.4, 3.5 E/p 27 The fourth, fifth and sixth terms of a geometric series are x, 3 and x + 8. 110 29 At the end of year 1, a company employs 2400 people. A model predicts that the number of employees will increase by 6% each year, forming a geometric series are x, 3 and x + 8. 110 29 At the end of year 1, a company employs 2400 people. sequence. a Find the predicted number of employees after 4 years, giving your answer to the nearest 10. (3) The company expects to expand in this way until the total number of employees first exceeds 6000 at the end of a year, N. b Show that (N - 1)log1.06. log 2.5 (3) c Find the value of N. 26 A geometric series has first term 4 and common ratio

r. The sum of the first three terms of the series is 7. - Sections 3.6, 3.7 (2) The company has a charity contribution exactly. d Given that the average employee charity contribution is £5 each year, find the total charity donation over the 10-year period from the end of year 1. Given that the average employee charity contribution is £5 each year, find the total charity donation over the 10-year period from the end of year 1. your answer to the nearest £1000. (3) \leftarrow Section 3.8 Review exercise 1 E/p 30 A geometric series is given by E 6 - 24x + 96x - 2 The series is convergent. a Write down a condition on x. ∞ Given that $\sum 6 \times (-4x) r = 1 r - 1$ (1) = 5 24 \leftarrow Section 3.5, 3.6 (5) \leftarrow Section 4.2 E 1 31 g(x) = $\sqrt{1 - x}$ a Show that the series expansion of g(x) up to and including the x3 term is x 3 x2 5 x3 1 + + + - (5) 2 8 16 b State the range of values of x for which the expansion is valid. (1) 35 h(x) = $\sqrt{4} - 9x$, |x| < 94 32 When (1 + ax)n is expanded as a series in ascending powers of x, the coefficients of x and x2 are -6 and 45 respectively. a Find the series expansion of h(x), in ascending powers of x, up to and including the x2 term. Simplify each term. (4) 1 b Show that, when x = 0, the exact $100 \sqrt{391}$ (2) value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(x) is 10 c Use the 4.1 p 3 (5) b Calculate the value of x. E 34 f(x) = (1 + x)(3 + 2x) - 3, |x| < 2 Find the binomial expansion of f(x) in ascending powers of x, up to and including the term in x3. Give each coefficient as a simplified fraction. b Find the coefficient as a simplified fraction. b Find the set of values of x for which the expansion of f(x) in ascending powers of x, up to and including the term in x3. 1000 (2) 3 c Substitute x = into the binomial 100 expansion in part a and hence obtain an approximation to $\sqrt{112}$. Give your answer to 5 decimal places. (3) 1 a Find the values of the constants a, b and c. 33 a Find the binomial expansion of 3 (1 + 4x) 2 in ascending powers of x up to Section 4.1 E value of (1 + 4x) 2 is $112 \sqrt{112}$ and including the x3 term, simplifying each term. (4) 3 b Show that, when x =, the exact 100 3 36 Given that (a + bx) -2 has binomial (4) b Find the coefficient of the x3 term in the expansion. (2) \leftarrow Section 4.2 E/p 3 + 5x 1 37 g(x) = ______, |x| < __3 (1 + 3x)(1 - x) Given that g(x) can be expressed in the A B form g(x) = ______. 3x 1 - x a find the values of A and B. (3) d Calculate the percentage error in your estimate to 5 decimal places. (2) b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the x2 term. Simplify each term. (6) \leftarrow Section 4.1 \leftarrow Sections 4.1, 4.3 111 Review exercise 1 E/p 3x - 1 A 1 B 382, |x| < 21 - 2x(1 - 2x)(1 - 2x)a Find the values of A and B. (3) 3x - 1b Hence, or otherwise, expand 2(1 - 2x)a in ascending powers of x, as far as the term in x3. Give each coefficient as a simplified fraction. (6) \leftarrow Sections 4.1, 4.3 E/p 25 39 f(x) = , |x| < 1(3 + 2x)2(1 - x) Challenge 1 The functions f and g are defined by f(x) = -3|x + 3| + 15, $x \in \mathbb{R}$ g(x) = -34x + 32, $x \in \mathbb{R}$ The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The circle C, with centre M, passes through points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The diagram shows a sketch of the graphs y = f(x) and y = g(x). + 3 + 2x(3 + 2x)21 - x a Find the values of A, B and C. A (4) b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in (6) x2. Simplify each term. \leftarrow Sections 4.1, 4.2, 4.3 E/p y C 4 x 2 + 30x + 31 B 40 = A + + x + 4 2x + 3 (x + 4)(2x + 3) a Find Α + the values of the constants A. B and C. (4) b Hence, or otherwise, expand $4 \times 2 + 31 \times + 30$ _ in ascending powers of (x + 4) (2x + 3) x, as far as the term in x2. Give each coefficient as a simplified fraction. (7) C P O B x y = g(x) a Find the equation of the circle. b Find the area of the triangle APB. \leftarrow Section 2.6 11 15 i=6 i=12 2 Given that an+1 = an + k, a1 = n and $\sum ai = \sum ai$, show that $n = \sum 2k$. \leftarrow Section 3.6 3 The diagram shows a sketch of the functions p(x) = |x2 - 8x + 12| and q(x) = |x2unit you should be able to: • Convert between degrees and radians and apply this to trigonometric graphs and their transformations - pages 114-116 • Know exact values of angles measured in radians - pages 117-118 • Find an arc length using radians - pages 118-122 • Find areas of sectors and segments using - pages 122-128 radians • Solve trigonometric equations in radians \rightarrow pages 128-132 \bigcirc Use approximate trigonometric values when θ is small \rightarrow pages 133-135 Prior knowledge check 1 Write down the exact values of the following trigonometric ratios. a cos 120° b sin 225° c tan (-300°) d sin (-480°) 2 \leftarrow Year 1, Chapter 10 Simplify each of the following expressions. a (tan θ cos θ)2 $\tan \theta \leftarrow \text{Year 1}$, Chapter 10 Show that a (sin $2\theta + \cos 2\theta$) $\equiv 1 + 2 \sin 2\theta \cos 2\theta 2 \cos 2\theta 2 b = -2 \sin \theta \equiv -2 \sin \theta = -2 \sin \theta =$ c cos20 3 $\sin\theta\cos\theta\sqrt{1}$ – distances between the pods around the \rightarrow Exercise 5B Q13 edge of a Ferris wheel. Solve the following equations for θ in the interval $0 < \theta < 360^\circ$, giving your answers to 3 significant figures where they are not exact. a 4 cos $\theta + 2 = 3$ c 6 tan 2 θ b 2 sin $2\theta = 1 + 10$ tan $\theta - 4 = \tan \theta d 10 + 5 \cos \theta = 12 \sin 2\theta \leftarrow$ Year 1, Chapter 5 5.1 Radian measure 1 So far you have probably only measured angles in degrees, with one degree representing ______ of a 360 complete revolution or circle. You can also measure angles in units called radians. 1 radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. If the arc AB has length r, then \angle AOB is 1 radian. Links A r O 1 rad r r B You always use radian measure when you are differentiating or integrating trigonometric functions. \rightarrow Sections 9.1, 11.1 Notation The circumference of a circle of radius r is an arc of length 2π r, so it subtends an angle of 2π radians at the centre of the circle. \blacksquare 2π radians = 360° \blacksquare π radians = 180° You can write 1 radian as 1 rad. $2\pi r \ 2\pi \ rad$ Hint This means that 1 radian = 57.295...° 180° I radian = π so multiply by $\pi 180^\circ \ 7\pi$ $\pi = 8 \times 180^\circ 82$ Convert the following angles into radians. Leave your answers in terms of π . a 150° π learn these important angles in radians: $\pi \equiv 30^\circ = _$ radians $\pi \equiv 60^\circ = _$ radians $3\pi \equiv 60^\circ = _$ radians $3\pi \equiv 60^\circ = _$ radians $3\pi \equiv 60^\circ = _$ radians $26\pi \equiv 45^\circ = _$ radians $3\pi \equiv 60^\circ = _$ radians $3\pi \equiv 60^\circ = _$ radians $3\pi \equiv 60^\circ = _$ radians $\pi \equiv 60^\circ = _$ radians $\pi \equiv 60^\circ = _$ radians $26\pi \equiv 45^\circ = _$ radians $360^\circ = 2\pi$ radians $3\pi \equiv 60^\circ = 2\pi$ radians $\pi \equiv
60^\circ = 2\pi$ -1 Use your calculator to evaluate trigonometric functions in radians. Watch out You need to make sure your calculator is in radians mode. c tan (2 rad) = -2.19 (2 d.p.) Example 4 Sketch the graph of y = sin x for $0 < x < 2\pi$. If the range includes values given in terms of π , you can assume that the angle has been given in radians. y 1π sin (__) = sin x for $0 < x < 2\pi$. $90^\circ = 12y = \sin x 0.5 \text{ O} - 0.5 \pi 2 \pi 3\pi 2 2\pi x - 1$ Example 5 Sketch the graph of $y = \cos (x + \pi)$ for $0 < x < 2\pi$. $y 1 y = \cos (x + \pi)$ The graph of $y = \cos (x + \pi)$ The graph of $y = \cos (x + \pi)$ a translation of the -a graph $y = \cos x 115$ Chapter 5 Exercise 5A 1 Convert the following angles in radians to degrees. $5\pi 5\pi \pi \pi a$ _ c _ _ d _ _ b _ 20 4 15 12 3π e _ 2 f 3π 2 Convert the following angles to degrees, giving your answer to 1 d.p. a 0.46 rad b 1 rad _ d $\sqrt{3}$ rad c 1.135 rad 3 Evaluate the following, giving your answers to 3 significant figures. _ b cos ($\sqrt{2}$ rad) a sin (0.5 rad) d sin (2 rad) e sin (3.6 rad) 4 Convert the following angles to radians, giving your answers as multiples of π. a 8° b 10° c 22.5° d 30° f 240° g 270° h 315° i 330° e 112.5° 5 Convert the following angles to radians, giving your answers to 3 significant figures. a 50° b 75° c 100° d 160° e 230° f 320° 6 Sketch the graphs of: a y = tan x for 0 < x < 2π b y = cos x for -π < x < π Mark any points where the graphs cut the coordinate axes. 7 Sketch the following graphs for the given ranges, marking any points where the graphs cut the coordinate axes. a $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi 2$ E/P by $= \cos 2x$ for $0 < x < 2\pi$. 3 $y = \sin (x - \pi)$ for $-\pi < x < \pi$. Problem-solving O x Make sure you write down the coordinates of all four points of intersection with the x-axis and the coordinates of the y-intercept. Write down the coordinates of the points at which the curve meets the coordinates of the y-intercept. Write down the coordinates of all four points at which the x-axis and the coordinates of the y-intercept. Write down the coordinates of the points at which the curve meets the coordinates of the y-intercept. $\sin = 4\sqrt{23} 2\pi$ $1 \equiv \cos = 42\sqrt{2}\pi\sqrt{1} \equiv \tan = 3\sqrt{2}\sqrt{\pi} \equiv \tan = 142$ You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle measured in radians (measured in radians acute angl $-\cos \theta \equiv \cos (\pi + \theta) = -\cos \theta \equiv \cos (2\pi - \theta) = \cos \theta \equiv \tan (\pi - \theta) = -\tan \theta = \tan \theta = \tan \theta = \tan \theta$ tan to find these results. \leftarrow Year 1, Chapter 10 6 Problem-solving Find the exact values of: $-7\pi 4\pi a \cos b \sin(b) 3 6$ You can also use the symmetry properties of $y = \cos x \cdot y = \cos x \cdot 1 \pi a 2 4\pi 3 4\pi 3 2\pi 3 \pi 0 3\pi 2 4\pi \pi \pi \cos b = -2 3 0 \pi \pi 3 2 \pi 3 \pi 2 2\pi x - 1 4\pi \pi$ is bigger than π . 3 3 Use $cos (n + \theta) = -cos \theta. 117 Chapter 5 b \pi 2 5\pi 6 \pi 0 7\pi - 6 3\pi 2 7\pi 5\pi sin (- _) = sin __ 6 6 7\pi 1 So sin (- _) = _ 6 2 Exercise \pi = sin __ 6 5\pi 5\pi 2\pi d cos __ e tan __ 11\pi 2\pi d cos __ e tan __ 1 tan _$ triangle ABC with AD = and BC = 2.3 D Show that DC = $k\sqrt{2}$, where k is a constant to be determined. 2 6 3 2 A π 3 B 5.2 Arc length Using radians greatly simplifies the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector. 118 r θ l Radians Example 7 Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 rad at the centre of the circle. Arc length = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Explore the arc length of a sector using GeoGebra. Use l = r θ , with r = $5.2 \times 0.8 = 4.16$ cm Explore the arc length of a sector using GeoGebra. Use l = 1.16 \times 0.16 \times 0.16 \times length of 2.45 cm. Find the angle $\angle AOB$ subtended by the arc at the centre of the circle. A 7cm O θ 2.45cm 7cm B l = 2.45 = 2.45 = 7 θ = 7 θ = Example r θ 7 θ Use l = r θ , with l = 2.45 and r = 7. θ 0.35 rad Using this formula gives the angle in radians. 9 An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ . r cm O A θ r cm B P = r θ + 2r = r(2 + θ) P So r = (2 + θ) Problem-solving When given a problem in words, it is often a good idea to sketch and label a diagram to help you to visualise the information you have and what you need to find. The perimeter = arc AB + OA + OB where arc AB = r0. Factorise. 119 Chapter 5 Example 10 C The border of a garden pond consists of a straight edge AB of length 2.4 m, and a curved part is an arc of a circle, centre O and radius 2 m. Find the length of C. 0 2m A O 2 A 2.4 m 2m B Online Explore the area of a sector using GeoGebra. x Problem-solving B 1.2 O Look for opportunities to use the basic trigonometric ratios rather than the more complicated cosine rule or sine rule. AOB is an isosceles triangle, so you can divide it into congruent right-angled triangles. Make sure your calculator is in radians mode. 1.2 sin $x = 2 \times 0.6435...$ rad Acute $\angle AOB = 2 \times 0.6435...$ 1.2870... rad So $\theta = (2\pi - 1.2870...)$ rad = 4.9961... rad So length of C = 2 θ . θ + acute $\angle AOB = 2\pi$ rad Exercise C = 2 θ . δ
+ acute $\angle AOB = 2\pi$ rad Exercise C = 2 θ . δ + acute $\angle AOB = 2\pi$ rad Exercise C = 2 θ . δ + acute $\angle AOB = 2\pi$ rad Exercise C = 2 θ . δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acute $\angle AOB = 2\pi$ rad Exercise C = 20. δ + acut b Find r when: i l = 10, θ = 0.6 ii l = 1.26, θ = 0.7 c Find θ when: i l = 10, r = 7.5 ii l = 4.5, r = 5.625 2 A minor arc AB of a circle, centre O and radius 10 cm, subtends an angle 5x at O. Find, in terms of π , the length of the minor arc AB. 3 iii r = 20, θ = π 8 5 iii l = 1.5 π , θ = π 12 V 3 Notation The minor arc AB is the shorter arc between points A and B on a circle. 3 An arc AB of a circle, centre O and radius 6 cm, has length 1 cm. Given that the chord AB has length 6 cm, find the value of l, giving your answer in terms of π. ____ 4 The sector of a circle of radius $\sqrt{10}$ cm contains an angle of $\sqrt{5}$ radians, as shown in the diagram Find the length of the arc, giving your answer in the form $p\sqrt{q}$ cm, where p and q are integers. 120 10 cm 5 rad 10 cm Radians P 5 2 cm Referring to the value of θ when the perimeter of the shaded region is 14 cm. Problem-solving θ The radius of the larger arc is 3 + 2 = 5 cm. 3 cm 2 cm P 6 A sector of a circle of radius r cm contains an angle of θ . E/P 8 In the diagram AB is the diameter of a circle, centre O and radius 2 cm. The point C is on the circumference such that $2 \angle COB = \pi$ radians. 3 a State the value, in radians, of $\angle COA$. C 2π 3 A O B 2 cm (1 mark) The shaded region enclosed by the chord AC, arc CB and AB is the template for a brooch. b Find the exact value of the perimeter of the brooch. P 9 (5) marks) The points A and B lie on the circumference of a circle with centre O and radius 8.5 cm. The point C lies on the major arc AB. E/P a Write down, in terms of R and r, the length of OC. (1 mark) b Using $\triangle OCE$, show that R sin $\theta = r (1 + \sin \theta)$. (3 marks) 3 c Given that sin θ and that the perimeter of the sector OAB is 4 21 cm, find r, giving your answer to 3 significant figures. (7 marks) P O 10 In the diagram OAB is a sector of a circle, centre C and radius r cm, touches the arc AB at T, and touches OA and OB at D and E respectively, as shown. 11 The diagram shows a sector AOB. The perimeter of the sector is twice the length of the arc AB. Find the size of angle AOB. R cm r cm D E C A B A T O B P 12 A circular Ferris wheel has 24 pods equally spaced on its circumference. 3n Given the arc length between each pod is m, and modelling each pod as a particle, 2 a calculate the diameter of the Ferris wheel. Given that it takes approximately 30 seconds for a pod to complete one revolution, b estimate the speed of the pod in km/h. 121 Chapter 5 E/P 13 The diagram above shows a triangular garden, PQR, with PQ = 12 m, PR = 7 m and $\angle QPR = 0.5$ radians. The curve SR is a small path separating the shaded patio area and the lawn, and is an arc of a circle with centre at P and radius 7 m. P 12 m Find: a the length of the path SR (2 marks) 7m S b the perimeter of the shaded patio, giving your answer to 3 significant figures. (4 marks) E/P 0.5 rad R Q 14 The shape XYZ shown is a design for an earring. X 15 mm 5 mm Y 5 mm O 1.1 rad Z The straight lines XY and XZ are equal in length. The curve YZ is an arc of a circle with centre O and radius 5 mm. The size of ∠YOZ is 1.1 radians and XO = 15 mm. a Find the size of ∠YOZ is 1.1 radians and XO = 15 mm. a Find the size of ∠YOZ is 1.1 radians and XO = 15 mm. a sector. To find the area A of a sector of a 1 circle use the formula A = __r2 θ , 2 where r is the radius of the circle and θ is the angle, in radians, contained by two radii and an arc. The smaller area is known as the minor sector and the larger is known as the major sector. Example 11 Find the area of the sector of a circle of radius 2.44 cm, given that the sector subtends an angle of 1.4 radians at the centre of the circle. 1 Area of sector = $x 2.442 \times 1.42 = 4.17 \text{ cm} 2$ (3 s.f.) 122 1 Use A = $r2 \theta$ with r = 2.44 and $\theta = 1.4$. 2 Radians Example 12 In the diagram, the area of the minor sector AOB is 28.9 cm2. Given that $\angle AOB = 0.8$ radians, calculate the value of r. 1 28.9 = r2 × 0.8 = 0.4r2 2 28.9 So r2 = $r2 \times 0.8 = 0.4r2 2 28.9$ So r2 = r2 + 0.2 Use the positive square root in this case as a length cannot be negative. Example 13 A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of a segment using GeoGebra. 55 m B Arc AB = 176 - (55 + 55) = 66 m $66 = 55\theta$ So $\theta = 1.2$ radians 1 Area of plot = $-\times 552 \times 1.22 = 1815$ m2 O Draw a diagram including all the data and let the angle of the sector be θ . Problem-solving In order to find the area of the sector, you need to know θ . Use the information about the perimeter is given, first find length of arc AB. Use the formula for arc length, $l = r\theta$. 1 Use the formula for area of a sector, $A = r2 \theta$. 2 You can find the area of a segment by subtracting the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a triangle: 2 2 1 1 $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2 \sin^2\theta$ for the area of a segment in a circle of radius r is $A = r^2\theta - r^2\theta$ 14 The diagram shows a sector of a circle. Find the area of the shaded segment. 7 cm 1.2 rad 7 cm Area of segment = 1 \times 72(1.2 - sin 1.2) 2 1 = \times 49 × 0.26796... 2 = 6.57 cm 2 (3 s.f.) Use A = 12 r2(\theta - sin \theta) with r = 7 and \theta = 1.2 radians. Make sure your calculator is in radians mode when calculating sin θ . Example 15 O 4m B 4m 5m A In the diagram above, OAB is a sector of a circle, radius 4 m. The chord AB is 5 m long. Find the area of the shaded segment. Calculate angle AOB first: $42 + 42 - 52 \cos 4 AOB =$ $2 \times 4 \times 47 = 32$ So $\angle AOB = 1.3502...$ Area of shaded segment $1 = \times 42(1.3502... - \sin 1.3502...)$ $21 = \times 16 \times 0.37448...$ 2 = 3.00 m2 (3 s.f.) 124 Problem solving In order to find the area of the segment you need to know angle AOB. You can use the cosine rule for a non-right-angled triangle. Watch out Use unrounded values in your calculations wherever possible to avoid rounding errors. You can use the memory function or answer button on your calculator. Radians Example 16 In the diagram, AB is the diameter of a circle of radius r cm, and $\angle BOC = \theta$ radians. Given that the area of $\triangle AOC$ is three times that of the shaded segment, show that $3\theta - 4 \sin \theta = 0$. C A r θ O r B Area of sector – area of triangle. 1 Area of segment = $r^2(\theta - \sin \theta) 2 1$ Area of $\triangle AOC = r^2 \sin(\pi - \theta) 2 1 = r^2 \sin \theta 2$ So So $\angle AOB = \pi$ radians. Area of $\triangle AOC = 3 \times area of shaded$ segment. Problems of $\theta = 3 \times r^2(\theta - \sin \theta) 3\theta - 4 \sin \theta = 0$ You might need to use circle theorems or properties when solving problems. The angle in a π semicircle is a right angle so $\angle ACB = 25D$ Exercise 1 Find the shaded area in each of the following circles. Leave your answers in terms of π where appropriate. a b c C C 0.6 rad π 6 8 cm 9 cm d e C π 1.2 cm 5 f 6 cm C C π 3 10 cm 1.5 rad 6 cm 7 π C 4 2 Find the shaded area in each of the following circles. cm C 125 Chapter 5 3 For the following circles with centre C, the
area A of the shaded sector is given. Find the value of x in each case. b a c A = 15 n cm 2 C 1.2 rad x cm C C n x cm 12 x rad 4.5 cm A = 20 cm 2 A = 12 cm 2 P 4 The arc AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and radius 6 cm. Find the area of the minor sector AOB. 5 The chord AB of a circle, centre O and ce circle, centre O and radius 10 cm, has length 18.65 cm and subtends an angle of θ radians at O. a Show that $\theta = 0.739$ radians (to 3 significant figures). b Find the perimeter of the sector. 7 The arc AB of a circle, centre O and radius r cm, is such that ∠AOB = 0.5 radians. Given that the perimeter of the minor sector AOB is 30 cm, a calculate the value of r b show that the area of the minor arc AB. π The arc AB of a circle, centre O and radius x cm, is such that angle AOB = radians. 12 Given that the arc length AB is 1 cm, 2 6l a show that the area of the sector can be written as π The area of the full circle is 3600 π cm². b Find the arca of the full circle is 3600 π cm². b Find the area of the shaded segment, show that $\theta + 2 \sin \theta = \pi$. 10 In the diagram, BC is the arc of a circle, centre O and radius 8 cm. The points A and D are such that AB O A 3 cm 5 cm O 1.6 rad D C 126 B Radians P 11 In the diagram, AB and AC are tangents to a circle, centre O and radius 3.6 cm. Calculate the area of the shaded region, 2π given that $\angle BOC = 0$ radians. $3 B A 3.6 \text{ cm } 2\pi 3 O C E/P 12$ In the diagram, AD and BC are arcs of circles with centre O, such that OA = OD = r cm, AB = DC = 8 cm and $\angle BOC = \theta$ radians. P P A r cm a Given that the area of the shaded region is 48 cm 2, show that $6 r = -4 (4 \text{ marks}) \theta b$ Given also that r = 10 θ , calculate the perimeter of the shaded region. (6 marks) D C A 14 The diagram shows a triangular plot of land. The sides AB, BC and CA have lengths 12 m, 14 m and 10 m respectively. The lawn is a sector of a circle with centre O, radius 10 cm, with ∠POQ = 0.3 radians. 4m C 14 m R Q 0.3 rad O S The point R is on OQ such that the ratio OR : RQ is 1 : 3. The region S, shown shaded in the diagram, is bounded by QR, RP and the arc PQ. 10 cm Find: E/P 6m Flowerbed B E/P B θ O 13 A sector of a circle of radius 28 cm has perimeter P cm and area A cm2. Given that A = 4P, find the value of P. a Show that ∠BAC = 1.37 radians, correct to 3 significant figures. 8 cm P a the perimeter of S, giving your answer to 3 significant figures. (6 marks) C 16 The diagram shows the sector OAB of a circle with centre O, radius 12 cm and angle 1.2 radians. B The line AC is a tangent to the circle with centre O, and OBC is a straight line. A The region R is bounded by the arc AB and the lines AC and CB. a Find the perimeter of DAB. R 1.2 rad 12 cm (5 marks) O 127 Chapter 5 P B 17 5 cm A 1.2 rad 0.3 rad C 5 cm D The diagram shows two intersecting sectors: ABD, with radius 5 cm and angle 1.2 radians, and CBD, with radius 12 cm and angle 0.3 radians. Find the area of the overlapping section. Challenge Find an expression for the area of a sector of a circle with radius r and arc length 1. 5.4 Solving trigonometric equations In Year 1, you learned how to solve trigonometric equations in the same way. Example 17 Find the solutions of these equations in the same way. Example 17 Find the solutions of these equations in the interval $0 < \theta < 2\pi$: a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta + 3 = 1$ a sin $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta = 0.3$ b 4 cos $\theta = 2$ c 5 tan $\theta = 0.3$ b 4 cos $\theta = 0.3$ b 4 co $3\pi 2 2\pi \theta$ Find the first value using your calculator in radians mode. sin $\theta = 0.3$ where the line y = 0.3 cuts the curve. Hence $\theta = 0.305$ rad or 2.84 rad (3 s.f.) 128 Since the sine curve is symmetrical in the interval $0 < \theta < \pi$, the second value is obtained by $\pi - 0.30469$... Radians b 4 cos $\theta = 21$ cos $\theta = 2\pi$ So $\theta = 3$ Watch out When the interval is obtained by $\pi - 0.30469$... given in radians, make sure you answer in radians. π S A First rewrite in the form cos $\theta = ...3 \pi$ Use exact values where possible. $3 \pi 3 T \pi$ So $\theta = _ 3$ or C $5\pi 3 \pi$ Putting __ in the four positions shown gives $35\pi \pi 2\pi 4\pi$ the angles __, ___, and ___ but cosine is only 3333 positive in the 1st and 4th quadrants. $\pi 5\pi \theta = 2\pi - _ = _ 33c5$ tan $\theta + _$ $3 = 15 \tan \theta = -2 \tan \theta = -0.4$ For the 2nd value, since we are working in radians, we use $2\pi - \theta$ instead of $360^\circ - \theta$. y y = tan θ - $\pi 2\pi 2\pi 3\pi 2$ Draw the graph of y = -0.4 1 1 cos θ = -2 where the line y = 2 cuts the curve. tan $-2(-0.4) = -2 \tan \theta = -2 \tan \theta = -2$ -0.3805... rad So $\theta = 2.76108...$ rad (2.76 rad to 3 s.f.) or $\theta = 5.90267...$ rad (5.90 rad to 3 s.f.) Use the symmetry and period of the tangent graph to find the required values. Watch out Always check that your final values are within the given range; in this case $0 < \theta < 2\pi$ (remember $2\pi \approx 6.283...$) 129 Chapter 5 Example 18 Solve the equation 17 cos $\theta + 2 \sin 2 \theta = 13$ in the interval $0 < \theta < 2\pi$. Problem-solving 17 cos $\theta + 3 \sin 2 \theta = 13$ 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2
$\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 $\theta - 17$ cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 cos $\theta + 3 - 3 \cos 2 \theta = 13$ 0 = 3 cos 2 \theta - 17 ies work the same in radians as in degrees. This is a quadratic so rearrange to make one side 0. If $Y = \cos \theta$, then $Y = \cos \theta$, then $Y = \cos \theta$ is between -1 and 1, so reject $\cos \theta = 5$. $y = \cos \theta 1$ Solve this equation to find θ . given in radians, answer in radians. O 0.841 π 2 π 3 π 2 -1 Second solution is $2\pi - 0.841068... = 5.442116...$ $\theta = 0.841$ or 5.44 (3 s.f.) 130 2π θ Radians Example 19 ____ 2, in the interval $0 < \theta < 2\pi$. Replace 3 θ by X and solve as normal. $\sqrt{3}$ ____ 2 As X = 3 θ , then the interval for X is $0 < X < 6\pi$ y 3. Always check that your solutions for θ are in the given interval for θ , in this case $0 < \theta < 2\pi$. 5E 1 Solve the following equations for θ , in the interval $0 < \theta < 2\pi$, giving your answers to 3 significant figures where they are not exact. a cos $\theta = -1.2$ Solve the following equations for θ , in the interval $0 < \theta < 2\pi$, giving your answers to 3 significant figures where they are not exact. giving your answers to 3 significant figures where they are not exact. $a 4 \sin \theta = 3 b 7 \tan \theta = 1 c 8 \tan$ 4 Solve the following equations for θ , giving your answers to 3 significant figures where appropriate, in the intervals indicated: $b 5 \sin \theta = 1$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = 0.02$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -7$, $0 < \theta < 4\pi f \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < 2\pi d 3 \cos \theta - 1 = -0.82$, $-\pi < \theta < -0.82$, $-\pi < 0.83$, $-\pi < 0.83$, $-\pi <$ the following equations for θ , in the interval $0 < \theta < 2\pi$, giving your answers to 3 significant figures where they are not exact. a 5 cos $2\theta = 4$ b 5 sin $3\theta + 3 = 1$ c $\sqrt{3}$ tan $4\theta - 5 = -4$ d $\sqrt{10}$ cos $2\theta + \sqrt{2} = 3\sqrt{2}$ $\sqrt{2}$ sin $3\theta - 1 = 0$, c 8 tan $2\theta = 7$, P $-\pi < \theta < 3\pi$ b 3 tan $2\theta = 2$ sin 2θ marks) 10 a Solve, for $-\pi < \theta < \pi$, (1 + tan θ)(5 sin $\theta - 2$) = 0. (4 marks) b Solve, for $0 < x < 2\pi$, 4 tan $x = 5 \sin x$. E (6 marks) 11 Find all the solutions, in the interval $0 < x < 2\pi$, 4 tan $x = 5 \sin x$. E (6 marks) 11 Find all the solutions, in the interval $0 < x < 2\pi$, 4 tan $x = 5 \sin x$. E (6 marks) 11 Find all the solutions, in the interval $0 < x < 2\pi$, 4 tan $x = 5 \sin x$. E (6 marks) 11 Find all the solutions, in the interval $0 < x < 2\pi$, 4 tan $x = 5 \sin x$. E (6 marks) E/P 7 12 Find, for $0 < x < 2\pi$, all the solutions of $\cos 2x - 1 = -\frac{1}{2} \sin 2x - 2$ giving each solution to 2 one decimal place. (6 marks) E/P 13 Show that the equation 8 sin x + 4 sin x - 20 = 4 has no solutions. E/P 14 a Show that the equation tan 2x - 2 tan x - 6 = 0 can be written as tan $x = p \pm \sqrt{q}$ where p and q are numbers to be found. (3 marks) (3 marks) b Hence solve, for $0 < x < 3\pi$, the equation tan 2x - 2 tan x - 6 = 0 can be written as tan $x = p \pm \sqrt{q}$ where p and q are numbers to be found. 0 giving your answers to 1 decimal place where appropriate. (5 marks) E/P 15 In the triangle ABC, AB = 5 cm, AC = 12 cm, $\angle ABC = 0.5$ radians and $\angle ACB = x$ radians. a Use the sine rule to find these values of x, giving your answer to 3 decimal places. (3 marks) Given that there are two possible values of x, b find these values of x, giving your answer to 3 decimal places. answers to 2 decimal places. 132 (3 marks) Radians 5.5 Small angle approximations You can use radians to find approximations for the values of sin θ , cos θ and tan θ . Notation When θ is small and measured in radians: • sin $\theta \approx \theta \theta 2$ • cos $\theta \approx 1 - 2$ You can see why these approximations work by looking at the graphs of $y = \sin \theta$, $y = \sin \theta$ $\cos \theta$ and $y = \tan \theta$ for values of θ close to 0. y In mathematics 'small' is a relative concept. Consequently, there is not a fixed set of numbers which are small and a fixed set of numbers which are small and a fixed set of numbers of θ . $2\theta \approx = (40) 21^{-1} - 2^{-1} -
2^{-1} - 2^{-1}$ $\sin 2\theta 2$ When θ is small, show that: $\sin 3\theta$ = 4 tan 20 3 a Find cos (0.244 rad) correct to 6 decimal places. b Use the approximation for cos 0 to find an approximate value for cos (0.244 rad). c Calculate the percentage error in your approximation. d Calculate the percentage error in the = $\theta \sin 4\theta 4\theta \tan 4\theta + \theta 2 c$ $= 4 + \theta 3\theta - \sin 2\theta \cos \theta - 1 \theta b$ approximation for $\cos 0.75$ rad. e Explain the difference between your answers to parts c and d. P E/P 4 The percentage error for $\sin \theta$ for a given value of θ is 1%. Show that $100\theta = 101 \sin \theta$. $4 \cos 3\theta - 2 + 5 \sin \theta 5$ a When θ is small, show that the equation can be written as $9\theta + 2$. (3 marks) $1 - \sin 2\theta 4 \cos 3\theta - 2 + 5 \sin \theta b$ when θ is small. 1 - sin 2θ 134 (1 mark) Radians Challenge 1 The diagram shows a right-angled triangle ABC. ∠BAC = θ. An arc, CD, of the circle with centre A and radius AC has been drawn on the diagram in blue. DB A θ C a Write an expression for the arc length CD in terms of AC and θ. Given that θ is small so that, $AC = AD \approx AB$ and $CD \approx BC$, b deduce that $\sin \theta \approx \theta$ and $\tan \theta \approx \theta$. 2 a Using the binomial expansion and ignoring terms in x4 and higher ______powers of x, find an approximation for $\sqrt{1 - x2}$, |x| < 1. $\theta \approx \theta = 0$. Mixed exercise P 5 1 Triangle ABC is such that AB = 5 cm, AC = 10 cm and ∠ABC = 90°. An arc of a circle, centre A and radius 5 cm, cuts AC at D. a State, in radians, the value of ∠BAC. b Calculate the area of the region enclosed by BC, DC and the arc A1, with centre M is drawn, with CD as diameter. A circular arc A2 with centre O and radius 17 cm, is drawn C from C to D. The shaded region R is bounded by the area of the triangle OCD (4 marks) b the area of the shaded region R. (5 marks) 3 The diagram shows a circle, centre O, of radius 6 cm. The points A and B are on the circumference of the shaded major sector is 80 cm2. Given that $\angle AOB = \theta$ radians, where $0 < \theta < \pi$, calculate: a the value, to 3 decimal places, of θ b the length in cm, to 2 decimal places, of the minor arc AB. (3 marks) (2 marks) O 17 cm 17 cm M D A2 R A A1 O B 135 Chapter 5 E/P 4 The diagram shows a sector OAB of a circle, centre O and radius r cm. The length of the arc AB is p cm and $\angle AOB$ is θ radians. a Find θ in terms of p and r. r cm (2 marks) 1 b Deduce that the area of the sector is pr cm2. (2 marks) 2 Given that r = 4.7 and p = 5.3, where each has been measured to 1 decimal place, find, giving your answer to 3 decimal places: E A c the least possible value of the area of the sector (2 marks) d the range of possible values of θ . (3 marks) θ O p cm r cm B 5 The diagram shows a circle centre O and radius 5 cm. The length of the minor arc AB is 6.4 cm. A a Calculate, in radians, the size of the acute angle AOB. (2 marks) The area of the minor sector AOB is R1 cm2 and the area of the shaded major sector is R2 cm2. b Calculate the value of R1. O (2 marks) E/P 6 The diagrams show the cross-sections of two drawer handles. Shape X is a rectangle ABCD joined to a semicircle with BC as diameter. The length AB = d cm and BC = 2d cm. Shape Y is a sector OPQ of a circle with centre O and radius 2d cm. Angle POQ is θ radians. Given that the areas of shape X (3 marks) a prove that $\theta = 1 + 2$ 4 Using this value of θ , and given that the areas of shape X and Y are equal, π (5 marks) a prove that $\theta = 1 + 2$ 4 Using this value of θ , and given that the areas of shape X and Y are equal, π (5 marks) a prove that $\theta = 1 + 2$ 4 Using this value of θ , and given that the areas of shape X are equal, π (5 marks) a prove that $\theta = 1 + 2$ 4 Using this value of θ , and given that $\theta = 1 + 2$ 4 Using this value of θ . perimeter of shape Y. (3 marks) d Hence find the difference, in mm, between the perimeters of shapes X and Y. E/P 7 The diagram shows a circle centre O and radius 6 cm. The chord PQ divides the circle into a minor segment R1 of area A1 cm2 and a major segment R2 of area A2 cm2. The chord PQ subtends an angle θ radians at O. a Show that A1 = $18(\theta - \sin \theta)$. (2 marks) π b show that $\sin \theta = \theta - 2$ (4 marks) (1 mark) P R2 R1 O 6 cm Given that A2 = 3A1, 136 Q Radians E/P 8 Triangle ABC + 3 cm, intersects AB and AC at P and Q respectively, as shown in the diagram. a Show that, to 3 decimal places, $\angle BAC = 1.504$ radians. E/P 9 A P 5 cm Q 9 cm (2 marks) b Calculate: C i the area, in cm2, of the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the sector APQ ii the area, in cm2, of the shaded region BPQC. 10 cm The diagram shows the sector APQ ii the area, in cm2, of the sector APQ ii the area, in cm2, of the sector APQ ii the area, in cm2, of th = 21/5. (2 marks) b Find, in cm, the perimeter of the sector OAB. (3 marks) R B r r The segment R, shaded in the diagram, is enclosed by the arc AB and the straight line AB. c Calculate, to 3 decimal places, the area of R. E/P 10 The shape of a badge is a sector ABC of a circle with centre A and radius AB, as shown in the diagram. The triangle ABC is equilateral and has perpendicular height 3 cm. a Find, in surd form, the length of AB. E O (2 marks) b Find, in terms of π, the area of the badge is (π + 6) cm. 3 11 There is a straight path of length 70 m from the point A to the point B. The points are joined also by a railway track in the form of an arc of the circle whose centre is C and whose radius is 44 m, as shown in the diagram. a Show that the size, to 2 decimal places, of ∠ACB is 1.84 radians. (2 marks) B C (2 marks) B C (2 marks) B C (2 marks) A Railway track Path A 44 m 44 m b Calculate: i the length of the radians. (2 marks) B C (2 marks) B C (2 marks) B C (2 marks) B C (2 marks) A Railway track Path A 44 m 44 m b Calculate: i the length of the radians. (2 marks) B C (2 bounded by the railway track and the path. P 12 The diagram shows the cross-section ABCD of a glass prism. AD = BC = 4 cm and both are at right angles to DC. AB is the arc of a circle, centre O and radius 6 cm. Given that $\angle AOB = 2\theta$ radians, and that the perimeter of the cross-section is $2(7 + \pi)$ cm, π a show that $(2\theta + 2 \sin \theta - 1) = 3\pi$ b verify that $\theta = 6$ c find the area of the cross-section. B 70 m C (6 marks) A 4 cm D B 4 cm C 6 cm O 20 6 cm 137 Chapter 5 P 13 Two circles C1 and C2, both of radius 12 cm have centres O1 and O2 respectively. O1 lies on the circumference of C1. The circles intersect at A and B, and enclose the region R. 2π a Show that $4AO1B = 2 \times 3 \times 50^{\circ} 1 \ 2$ Area = $2r \theta 1 \ 2 = 2 \times 3 \times 50 \ 3$ cm = 225 cm2 a Identify the mistake made by the student. (1 mark) b Calculate the correct area of the sector. (2 marks) 15 When θ is small, find the approximate values of: $\cos \theta - 1$ a $\theta \tan 2\theta 2(1 - \cos \theta) - 1$ b $\tan \theta - 17 + 2 \cos 2\theta$ b Hence write down the value of is small. tan $2\theta + 3 E/P$ (3 marks) (1 mark) 17 a When θ is small, show that the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3
marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solutions of the equation (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 203\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 200\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 200\theta + 15 = 0$ b Hence, find the solution (3 marks) $40\theta 2 - 200\theta + 15 = 0$ b Henc approximate value of $\cos 4\theta - \sin 4\theta$. 19 Solve the following equations for θ , giving your answers to 3 significant figures where appropriate, in the intervals indicated. a 3 sin $\theta = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$, $0 < \theta < \pi 1$ c tan $\theta + = 2$ and $y = 3 \cos x$ on the same axes ($0 < x < 2\pi$), marking on all the points where the graphs cross the axes. b Write down how many solutions there are in the given range for the equation 5 sin $x = 3 \cos x$. c Solve the equation 5 sin $x = 3 \cos x$. c Solve the equation 5 sin $x = 3 \cos x$. $-\theta$) as a single trigonometric function. 2 π b Hence solve 4 sin θ - cos (_ $-\theta$) = 1 in the interval $0 < \alpha < 2\pi$. Give your answers to 2 3 significant figures. (3 marks) 3π 22 Find the values of x in the interval $0 < x < _$ which satisfy the equation 2 sin 2x + 0.5 _____ = 2 1 - sin 2x 23 A teacher asks two students to solve the equation 2 cos 2x = 1equation. E/P (6 marks) 25 a Show that the equation 5 sin $x = 1 + 2 cos^2 x$ can be written in the form 2 sin x - 3 = 0 Solve the equation yourself then compare your working with the student's answer. (1 marks) (2 marks) ($\sin 2x + 9\cos x - 6 = 0$ can be written in the form $(1 - 5\cos 2x)$ b Hence solve, for $0 < x < 4\pi$, $4\sin 2x = 5\sin 2x$ giving your answers to 1 decimal place. E/P (6 marks) b Hence solve, for $0 < x < \pi$, $\tan 2x = 5\sin 2x$ giving your answers to 1 decimal place. E/P (6 marks) b Hence solve, for $0 < x < \pi$, $\tan 2x = 5\sin 2x$ giving your answers to 1 decimal place. E/P (6 marks) b Hence solve, for $0 < x < \pi$, $\tan 2x = 5\sin 2x$ giving your answers to 1 decimal place. your answers to 1 decimal place where appropriate. You must show clearly how you obtained your answers. E π 28 a Sketch, for $0 < x < 2\pi$, the graph meets the coordinate axes. c Solve, for $0 < x < 2\pi$, the equation π y = cos (x +) = 0.65, 6 giving your answers in radians to 2 decimal places. Challenge Use the small angle approximations to determine whether the following equations have any solutions close to $\theta = 0$. In each case, state whether each root of the resulting quadratic equation is likely to correspond to a solution of the original equation. a 9 sin θ tan $\theta + 25$ tan $\theta = 6$ b 2 tan $\theta + 3 = 5 \cos 4\theta$ c sin $4\theta = 37 - 2\cos 2\theta \ 140 \ (5 \ marks) \ (2 \ marks) \ (3 \$ trigonometric ratios of these angles measured in radians. $\pi 1 \cdot \sin = 62 \sqrt{3} \pi$ $\cdot \sin = 32 \sqrt{2} \pi - 1$ $\cdot \sin = 4\sqrt{2} 2 \pi \sqrt{3} \cdot \cos = 62\pi 1 \cdot \cos = 4\sqrt{2} 2 \pi \sqrt{3} \pi 1$ $= \cdot \tan = 6\sqrt{3} \pi \cdot \tan = 1434$ You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ . • sin $(\pi - \theta) = -\cos \theta + \theta + \cos (\pi - \theta) = -\cos \theta + \theta + \cos (\pi - \theta) = -\sin \theta + \sin (\pi - \theta) = -\sin (\pi$ 5 To find the arc length l of a circle use the formula l = rθ, where r is the radius of the circle and θ is the angle, in radians, contained by the sector. r θ A 7 The area of a segment in a circle of radius r is $A = 12 r^2(\theta - \sin \theta) 8$ When θ is small and measured in radians: • sin $\theta \approx \theta \cdot \theta 2$ • cos $\theta \approx 1 - 2141 6$ Trigonometric functions of secant, cosecant and cotangent and their relationship to cosine, sine and tangent \rightarrow pages 143-145 Understand the graphs of secant, cosecant and cotangent and their \rightarrow pages 145-149 domain and range \oplus Simplify expressions, prove simple identities and solve equations \rightarrow pages 145-149 domain and use inverse the solutions in the interval 0 ø x ø 2n to the equation 3 sin 2(2x) = 1. - Section 5.5 142 Trigonometric functions and resonance in bridges. You will use the functions and resonance in bridges. (sec), cosecant (cosec) and cotangent (cot) are known as the reciprocal trigonometric functions. $1 \\ \blacksquare$ sec x =______ (undefined for values of x for which sin x = 0) $\sin x 1 \\ \blacksquare$ cosec x =______ tan x (undefined for values of x for which tan x = 0) You can also write cot x in terms of sin x sin x Example 1 Use your calculator to write down the values of: a sec 280° b cot 115° 1 a sec 280° = $= 5.76 (3 \text{ s.f.}) \cos 280° 1 \text{ b cot } 115° =$ = -0.466 (3 s.f.) tan 115° Example Make sure your calculator is in degrees mode. 2 Work out the exact values of: 3π b cosec 4 a sec 210° Exact here means $cos 210^\circ y S A 30^\circ 30^\circ T O x C$ $\sqrt{3} \sqrt{3} cos 30^\circ =$ $so -cos 30^\circ =$ 222 So sec $210^\circ = -\sqrt{3} 210^\circ$ is in 3rd quadrant, so $cos 210^\circ = -cos 30^\circ$. $2\sqrt{3} O r sec 210^\circ =$ $-\frac{14}{3} Sin() 4 y S \pi 3\pi 3\pi$ is in the 2nd $4 x O T C 3\pi 1$ So cosec = $4 sin() 4 \pi 1$ sin() = $4\sqrt{2}$ 3π So cosec () $= \sqrt{2} 4$ Exercise 6A 1 Without using your calculator, write down the sign of the following trigonometric ratios. a sec 300° b cosec 190° d cot 200° e sec 95° c
cot 110° 2 Use your calculator to find, to 3 significant quadrant, so sin $__$ = +sin $_$ 4 3 π 4 4 A π 4 4 x O T C 3 π 1 So cosec $__$ = $_$ figures, the values of: a sec 100° d cot 550° 11π g cosec _____10 b cosec 260° 4π e cot ____3 h sec 6 rad c cosec 280° f sec 2.4 rad 3 Find the exact values (in surd form where appropriate) of the following: a cosec 90° b cot 135° c sec 180° d sec 240° e cosec 300° f cot(-45°) g sec 60° 4π j cot ____3 h cosec (-210°) 11π k sec ____) 4 P 4 Prove that cosec $(\pi - x) \equiv \text{cosec } x$. P 5 Show that cot 30° sec 30° = 2. P $2\pi 2\pi 6$ Show that cosec $+ \sec = a + b\sqrt{3}$ where a and b are real numbers to be found. 3 3 144 4 Trigonometric functions Challenge The point P lies on the unit circle, centre O. The radius OP makes an acute angle of θ with the positive x-axis. The tangent to the circle at P intersects the coordinate axes at points A and B. y A P 1 O θ B x Prove that a OB = sec θ b OA = cosec θ c AP = cot θ 6.2 Graphs of y = cos x, y = sin x and y = tan x to sketch the graphs of their reciprocal functions. Example 3 Sketch, in the interval -180° θ θ 180°, the graph of y = cos x, y = sin x and y = tan x to sketch the graphs of their reciprocal functions. sec θ . v First draw the graph v = cos θ . v = sec θ For each value of θ , the value of sec θ is the reciprocal of the corresponding value of cos θ . In particular: cos $0^\circ = 1$, so sec $180^\circ = -1$. 1 - 180 - 90 O -1 90 y = cos θ 180 θ As θ approaches 90° from the left, cos θ is +ve but approaches zero, and so sec θ is +ve but becomes increasingly large. At $\theta = 90^\circ$, sec θ is undefined and there is a vertical asymptote. This is also true for $\theta = -90^\circ$. As θ approaches 2ero, and so sec θ is -ve but becomes increasingly large negative. 145 Chapter 6 or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$. $y = \sec x y 1 - 450^\circ - 5\pi 2 - 270^\circ - 3\pi 2 - 90^\circ - \pi 2 O - 1$ Notation $90^\circ 270^\circ 450^\circ \pi 2 3\pi 2 5\pi x 2$ The domain can also be given as $(2n + 1)\pi x \in \mathbb{R}$, $x \neq _$ ______, $n \in \mathbb{Z} 2 \mathbb{Z}$ is the symbol used for integers, i.e. positive and negative whole numbers including 0. • The domain of y = sec x is $x \in \mathbb{R}$, $x \neq 90^\circ$, 270° , 450° , ... or any odd multiple of $90^\circ \pi 3\pi 5\pi \pi \bullet$ In radians the domain is $x \in \mathbb{R}$, $x \neq _$ ______, ... or any odd multiple of $y = \sec x$ is $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$. $y = \csc x y 1 - 10^\circ y \downarrow 1$ $360^\circ - 180^\circ - 2\pi - \pi 180^\circ 360^\circ \pi 2\pi O x - 1$ Notation The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0^\circ$, 180° , ... or any multiple of $\pi \cdot \operatorname{The}$ domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0^\circ$, 180° , ... or any multiple of $180^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ$, 180° , ... or any multiple of $180^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ$, $180^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ$, $180^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ$, $180^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$, $x \neq 0^\circ \cdot \operatorname{In}$ and $x \in \mathbb{R}$. π radians. It has vertical asymptotes at all the values of x for which tan x = 0. y = cot x y 146 - 360° - 180° - 2π - π 1 O - 1 180° 360° π 2π x Trigonometric functions • The domain of y = cot x is x ∈ \mathbb{R} , x ≠ 0°, 180°, 360°,... or any multiple of 180° • In radians the domain is x ∈ \mathbb{R} , x ≠ 0, π, 2π,... or any multiple of π • The range of y = cot x is y ∈ \mathbb{R} Example Notation The domain can also be given as $x \in \mathbb{R}$, $x \neq n\pi$, $n \in \mathbb{Z}$. 4 a Sketch the graph of y = 4 cosec x, $-\pi \phi x \phi \pi$. b On the same axes, sketch the line y = x. c State the number of solutions to the equation 4 cosec x, $-\pi \phi x \phi \pi$. b On the same axes, sketch the line y = x. c State the number of solutions to the equation 4 cosec x, $-\pi \phi x \phi \pi$. b On the same axes, sketch the line y = x. c State the number of solutions to the equation 4 cosec x - x = 0. You only need to draw the graph for $-\pi < x < \pi$. y = 4 cosec x = x y = 4 cosec and y = g(x). 5 Sketch, in the interval $0 \ \theta \ \theta \ 360^\circ$, the graph of $y = 1 + \sec 2\theta$. y Online $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graphs of reciprocal trigonometric functions using technology. 1 O y θ Step 1 Draw the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 270^\circ\ 360^\circ\ x$ Explore transformations of the graph of $y = \sec \theta - 1\ 90^\circ\ 180^\circ\ 1$ $90^{\circ} 135^{\circ} 180^{\circ} 225^{\circ} 270^{\circ} 315^{\circ} 360^{\circ} \theta - 1 y = 1 + \sec 2\theta y$ Step 3 0 Translate by the vector (). 1 2 O 45^{\circ} 90^{\circ} 135^{\circ} 180^{\circ} 225^{\circ} 270^{\circ} 315^{\circ} 360^{\circ} \theta = \cot \theta 2 a Sketch, on the same set of axes, in the interval $-\pi \phi x \phi \pi$, the graphs of $y = \cot x$ and y = -x. b Deduce the number of solutions of the equation $\cot x + x = 0$ in the interval $-\pi \phi x \phi \pi$. 3 a Sketch, on the same set of axes, in the interval $0 \phi \theta \phi 360^\circ$, the graphs of $y = \cot \theta$ and $v = \sin 2\theta$. b Deduce the number of solutions of the equation $\cot \theta = \sin 2\theta$ in the interval $0 \ \theta \ \theta \ 360^\circ$. 5 a Sketch on separate axes, in the interval $0 \ \theta \ \theta \ 360^\circ$, the graphs of $y = \tan \theta$ and $y = \cot(\theta + 90^\circ)$. b Hence, state a relationship between $\tan \theta \ and \ cot(\theta + 90^\circ)$. 148 Trigonometric functions P 6 a Describe the relationships between the graphs of: $\pi i y = \tan(\theta + \alpha)$ and $y = \cot(-\theta)$ and $y = \cot(-\theta)$ and $y = \cot(-\theta)$ and $y = \cot(-\theta)$, 360° , the graphs of: by = -cosec θ ay = sec 2θ dy = cosec $(\theta - 30^\circ)$ ey = 2 sec $(\theta - 60^\circ)$ cy = 1 + sec θ fy = cosec $(2\theta + 60^\circ)$ hy = 1 - 2 sec θ gy = -cot (2θ) In each case show the coordinates of any maximum and minimum points, and of any points at which the curve meets the axes. 8 Write down the periods of the following functions. Give your answers in terms of π . b cosec 2 θ 1 a sec 3 θ E/P c 2 cot θ d sec($-\theta$) 9 a Sketch, in the interval $-\pi \varphi \theta \varphi 2\pi$, the graph of y = 3 + 5 cosec x = k has no solutions. E/P 10 a Sketch the graph of y = 1 + 2 sec θ in the interval $-\pi \varphi \theta \varphi 2\pi$. b Write down the θ -coordinates of , and give the smallest 1 + 2 sec θ positive values of θ at which they occur. (3 marks) (2 marks) (2 marks) (4 marks) 6.3 Using sec x, cosec x and cot x You need to be able to simplify expressions, prove identities and solve equations involving sec x. points at which the gradient is zero. 1 c Deduce the maximum and minimum values of cosec x and cot x. \blacksquare sec x = k and cosec x = k have no solutions for -1 < k < 1. Example 6 Simplify: a sin θ cot θ sec θ b sin θ cos θ (sec θ + cosec θ) 149 Chapter 6 a sin
θ cot θ sec θ 1 1 cos θ 1 = sin $\theta \times$ 1 sin θ 1 cos θ = 1 1 1 b sec θ + cosec θ = _____ + ____ cos θ sin θ sin θ + cos θ = _____ $\sin \theta \cos \theta$ So $\sin \theta \cos \theta$ (sec θ + cosec θ) = $\sin \theta + \cos \theta$ Example Write the expression in terms of sin and \cos , $\cos \theta + \sin \theta$ and $\cos \theta = -\cos \theta$ and $\sec \theta = -\cos \theta = -\cos \theta$ and $\sec \theta = -\cos \theta =$ $\sec 2 \theta$ $\sin \theta \sin \theta \sin 2 \theta$ The denominator $\sec 2 \theta + \csc 2 \theta + 1 \equiv$ θ has no solutions. sec 2 θ + cosec 2 θ a Consider LHS: The numerator cot θ cosec θ cos θ cos θ 1 = + cosec 2 θ cot θ cosec θ b Hence explain why the equation $\cos 2\theta \sin 2\theta 1 \equiv$ cos2 θ $\cos 2\theta \sin 2\theta \sin 2\theta + \cos 2\theta \equiv$ $\equiv (sin 2 \theta) (cos 2 \theta sin 2 \theta) cos \theta$ $cos 2 \theta sin 2 \theta] = x 2 1 sin \theta Write the expression in terms of sin and cos, cos <math>\theta 1$ using $cot \theta \equiv$ and $cose c \theta =$ $sin \theta sin \theta Write the expression in terms of sin and cos, 1 2 1 using sec 2 <math>\theta \equiv$ $\sin 2 \theta \cot \theta \csc \theta So$ $\sec 2 \theta + \csc 2 \theta \cos \theta 1 \div$ and cos $\theta \cos 2 \theta = \frac{1}{2} \sin 2 \theta$ Remember that $\sin 2 \theta + \cos 2 \theta = 1$. Remember to invert the fraction when changing from $\div \sin \theta$ invert the fraction when changing from $\div \sin \theta$ is $\theta = 1$. Remember to invert the fraction when changing from $\div \sin \theta = 1$. Remember to invert the fraction when changing from $\div \sin \theta = 1$. Remember to invert the fraction when changing from $\div \sin \theta = 1$. Remember to invert the fraction when changing from $\div \sin \theta = 1$. Remember to invert the fraction when changing from $\div \sin \theta = 1$. The fraction whe y = cos x 1 113.6° 246.4° 90° 180° 270° 360° θ O -0.4 Sketch the graph of y = cos x for the given interval. The graph is symmetrical about θ = 180°. Find the principal value using your calculator then subtract this from 360° to find the second solution. You could also find all the solutions using a CAST diagram. This method is shown for part b below. - $1 \theta = 113.6^\circ$, 246.4° = 114°, 246° (3 s.f.) 1 b = 0.6 tan 20 5 1 tan 20 = = 0.6 3 Let X = 20, so that you are solving 5 tan X = , in the interval $0 < X < 720^\circ$. 3 S A 59.0° 59.0° T Calculate angles from the diagram. 1 Substitute for cot 20 and then simplify to tan 20 get an equation in the form tan 20 = k. Draw the CAST diagram, with the acute angle X = tan -1 _35 drawn to the horizontal in the 1st and 3rd quadrants. C X = 59.0°, 239.0°, 419.0°, 599.0° So θ = 29.5°, 120°, 210°, 300° (3 s.f.) Remember that X = 2 θ . 151 Chapter 6 6C Exercise 1 Rewrite the following as powers of sec θ or cot θ . 1 1 4 c ______a ____b ____3 6 sin θ 2 cos2 θ tan θ sec θ e ______cos4 θ _______ $\cos \theta = \sin \theta = \cos \theta \cot \theta P 5$ Solve, for values of θ in the interval $0 < \theta < 360^\circ$, the following equations. Give your answers to 3 significant figures where necessary. a sec $\theta = -3 - 4 = 0$ f $5 \cos \theta = 3 \cot \theta c 5 \cot \theta = -2 \frac{1}{9} \cot 2 \theta - 8 \tan \theta = 0$ d $\cos c \theta = 2 \ln 2 \sin \theta = \csc \theta 6 \frac{1}{9} \sin \theta = 100$ f θ in the interval -180° $\theta < 180^\circ$, the following equations: a cosec $\theta = 1$ d 2 cosec $\theta = 0$ g cosec $\theta = 0$ g cosec $2\theta = 4$ P d $(1 - \cos x)(1 + \sec x) = \sin x \tan x$ b sec $\theta = 2 \cos \theta$ h 2 cos $\cos 2\theta = 152$ $2\sqrt{3}$ $3b \cot \theta = -\sqrt{3}$ $3\pi \pi d \sec \theta = \sqrt{2} \tan \theta$ ($\theta \neq$ $0 \neq$) 2 2 Trigonometric functions E/P 8 In the diagram AB = 6 cm is the solving AB is the diameter of the circle, so $\angle ACB = 90^\circ$. cosec x - cot x 9 a Prove that = cosec x. 1 - cos x E/P C A θ 6 cm B (4 marks) cosec x - cot x b Hence solve, in the interval $-\pi < x < \pi$, the equation = 2.1 - cos x E/P D (4 marks) b Given that CD = 16 cm, calculate the length of the chord AC. E/P T sin x tan x 10 a -1 = - has no solutions. $21 - \cos x \sin x \tan x 1$ b Hence explain why the equation -1 = - has no solutions. $21 - \cos x 1 + \cot x 11$ Solve, in the interval $0 < x < 360^\circ$, the equation $= 5.1 + \tan x (3 \text{ marks}) (4 \text{ marks}) (1 \text{ mark}) (8 \text{ marks})$ roblem-solving 1 Use the relationship cot x = -1 to form a quadratic tan $x \leftarrow \text{Year } 1$, Prove that Section 10.5 equation in tan x. 6.4 Trigonometric identities You can use the identity $\sin 2x + \cos 2x \equiv 1$ to prove the following identities. $\blacksquare 1 + \tan 2x \equiv \sec 2x$ $\blacksquare 1 + \cot 2x \equiv \sec 2x \equiv 1 + \cot 2x \equiv \sec 2x$ Prove that $1 + \cot 2x \equiv \csc 2x$. 153 Chapter 6 a Unless otherwise stated, you can assume the identity $\sin 2x + \cos 2x \equiv 1$ in proofs involving $\csc x = 1 \sin 2x$ $\cos 2x \cos 2x + 1 \equiv \cos 2x \cos 2x + 1 \equiv (\cos x) \sin x$ b 2 1 so $1 + \tan 2x \equiv \sec 2x \sin 2x + \sin 2x$ $\sin 2x = \sin 2x$ = sin 2 x sin 2 x sin 2 x sin x 1 Use tan x = cos x and sec x = cos x Divide both sides of the identity by sin 2 x. cos x 2 1 2 1 + (___) = (___) sin x sin x so Divide both sides of the identity by cos 2 x. 1 + cot 2 x = cosec 2 x cos x 1 Use cot x = _____ and cosec x = _____ sin x sin x sin x sin x sin x sin x for the identity by cos 2 x. 1 + cot 2 x = cosec 2 x cos x 1 Use cot x = ______ and cosec x = ______ sin x sin $\cos 2 \ge 2 \equiv 1 \cos 2 \ge 1 + 1$ $(\cos e^2 \theta - \cot 2 \theta) \equiv \csc 2 \theta + \cot 2 \theta \cos 2 \theta 1 \equiv - + 2 \sin \theta \sin 2 \theta 1 + \cos 2 \theta \equiv \sin 2 \theta + \sin 2 \theta \sin 2 \theta = \sin 2 \theta + \sin 2
\theta \sin 2 \theta = \sin 2 \theta + \sin 2 \theta$ $\cos \theta = 1$, $\cos^2 \theta = 1$ can start from either the LHS or the RHS when proving an identity. Try starting with the LHS using $\cos 2\theta = 1 - \sin 2\theta$ and $\sec 2\theta = 1 - \sin 2\theta$ and $\sec 2\theta = 1 - \sin 2\theta$ and $\sec 2\theta = 1 - \sin 2\theta$. The equation can be rewritten as $4(1 + \cot 2\theta) - 9 = \cot \theta$ So $4 \cot 2\theta - \cot \theta - 5 = 0$ ($4 \cot \theta - 5$)($\cot \theta + 1$) = 0 So \therefore This is a quadratic equation. You need to write it in terms of one trigonometrical function only, so use $1 + \cot 2\theta = \csc 2\theta$. Factorise, or solve using the quadratic formula. 5 cot $\theta = -1.4$ for tan $\theta =$ horizontal is $\tan -1$ $45 = 38.7^\circ$. 38.7° T C If α is the value the calculator gives for $\tan -1$ 45° , then the solutions are α and $(180^\circ + \alpha)$. $\theta = 38.7^\circ$, 210° (3 s.f.) For $\tan \theta = -1$ S As $\tan \theta$ is $-\nu e$, θ is in the 2nd and 4th quadrants. The acute angle to the horizontal is $\tan -1$ $1 = 45^\circ$. A 45° If α is the value the calculator gives for $\tan -1$ (-1), then the solutions are $(180^\circ + \alpha)$ and $(360^\circ + \alpha)$, as α is not in the given interval. 45° C T Online Solve this equation numerically using your calculator. $\theta = 135^\circ$, 315° Exercise 6D Give answers to 3 significant figures where necessary. 1 Simplify each of the following expressions. a $1 + \tan 2 \theta$ b (sec $\theta - 1$)(sec $\theta + 1$) tan θ sec θ g $1 + \tan 2 \theta h (1 - \theta)$ $\sin 2\theta (1 + \tan 2\theta) 1 d (\sec 2\theta - 1) \cot \theta e (\csc 2\theta - \cot 2\theta) 2 j (\sec 4\theta - 2 \sec 2\theta \tan 2\theta + \tan 4\theta) P c \tan 2\theta (\csc 2\theta - 1) f 2 - \tan 2\theta + \sec 2\theta \csc \theta \cot \theta i$ 1 + $\cot 2\theta k 4 \csc 2\theta + 4 \csc 2\theta \cot 2\theta k 2$ Given that $\csc x =$ cosec x, where k > 1, find, in terms of k, possible values of $\cot x$. 3 Given that $\cot \theta = -\sqrt{3}$, and that 90° < θ < 180°, find the exact values of: a sin θ b cos θ 4 Given that tan θ = 4, and that 180° < θ < 270°, find the exact values of: 3 a sec θ b cos θ c sin θ 5 Given that tos θ = 25, and that θ is a reflex angle, find the exact values of: 24 a tan θ 156 b cosec θ Trigonometric functions P 6 Prove the following identities. a sec 4 θ - tan 4 θ = sec2 θ + tan2 θ b cosec2 x - sin2 x = cot2 x + cos2 x c sec2 A(cot2 A - cos2 A) = cot2 A d 1 - cos2 θ = (sec2 θ - 1)(1 - sin2 θ) 1 - tan2 A = 1 - 2 sin2 A e ______1 + tan2 A f sec2 θ = sec2 θ cosec2 θ g cosec A sec2 A = cosec A + tan A sec A h (sec θ - sin θ) (sec θ + sin θ) = tan2 θ + cos2 θ P 7 Given that 3 tan2 θ + 4 sec2 θ = 5, and that θ is obtuse, find the exact value of sin θ . P 8 Solve the following equations in the given intervals. a sec2 θ = 3 tan θ , $0 < \theta < 360^\circ$ b tan2 θ - 2 sec θ + 1 = 3 cot θ , $-180^\circ < \theta < 180^\circ$ d cot θ = 1 - cosec2 θ , $0 < \theta < 360^\circ$ 1 1 f (sec θ - cos θ)2 = tan θ - sin 2 θ , $0 < \theta < \pi$ cosec2 θ = 3 tan θ , $0 < \theta < \pi$ cosec2 θ = 3 tan θ , $0 < \theta < 360^\circ$ b tan2 θ - 2 sec θ + 1 = 3 cot θ , $-180^\circ < \theta < 180^\circ$ d cot θ = 1 - cosec2 θ , $0 < \theta < 360^\circ$ 1 1 f (sec θ - cos θ)2 = tan θ - sin 2 θ , $0 < \theta < \pi$ g tan2 $2\theta = \sec 2\theta - 1, 0 < \theta < 180^\circ$ E/P h sec $2\theta - (1 + \sqrt{3})$ tan $\theta + \sqrt{3} = 1, 0 < \theta < 2\pi$ 9 Given that tan $2k = 2 \sec k$, a find the value of sec k (4 marks) b deduce that cos $k = \sqrt{2} - 1$. c Hence solve, in the interval $0 < k < 360^\circ$, 1 decimal place. E/P (2 marks) tan $2k = 2 \sec k$, giving your answers to 10 Given that $a = 4 \sec x$, $b = \cos x$ and $c = \cot x$, a express b in terms of a 16 b show that c 2 = 2a - 16 E/P E/P (3 marks) 11 Given that $x = \sec \theta + \tan \theta$, 1 a show that $x = \sec \theta - \tan \theta$. 1 b Hence express $x^2 + 2 + 2$ in terms of θ , in its simplest form. x p - 1 12 Given that $2 \sec^2 \theta - \tan^2 \theta = p$ show that $\cos^2 \theta = 2a - 16 E/P E/P (3 marks) (3 marks) (3 marks) (3 marks) (5 marks$ Chapter 6 6.5 Inverse trigonometric functions You need to understand and use the inverse trigonometric functions arcsin x, arccos x and arctan x and their graphs. that the inverse function exists since $y = \sin x$ is a one-to-one function for the restricted domain. Only one-to- Section 2.3 one functions have inverses. $y = \arcsin x + 3$ and y coordinates of $\pi = -3$ arcsin x + 3 and y coordinates of $\pi = -3$ arcsin x + 3 and y coordinates of $\pi = -3$ arcsin x + 3 a points interchange when reflecting in y = x. For example: $\pi \pi$ (2, 1) \rightarrow (1, 2) Trigonometric functions \blacksquare The inverse function of cos x is -1 < x < 1. \bullet The arccos x (2, 1) \rightarrow (1, 2) Trigonometric functions \blacksquare The inverse function of tan x is called arctan x. y π 2 y = arctan x x O Watch out Unlike arcsin x and arccos x, the function arctan x is defined for all real values of x. $-\pi$ 2 The domain of y = arctan x is x $\in \mathbb{R}$. $\pi \pi \oplus$ The range of y = arctan x < or -90° < arctan x < 90° . 2 2 \oplus Example 14 Work out, in radians, the values of: a arcsin(-2) $\sqrt{2}$ b arccos(-1) a c arctan($\sqrt{3}$) $\pi 2 S A \pi 4 T - \pi 2 \pi \arcsin(-) = -24 \sqrt{2}$ C $\pi \pi$ You need to solve, in the interval $- < x < ..., 22 \sqrt{2}$ the equation sin $x = -... 2 \pi$ The angle to the horizontal is and, as sin is -ve, $\overline{4}$ it is in the 4th quadrant. Online Use your calculator to evaluate inverse trigonometric functions in radians. 159 Chapter 6 by 1 y = cos x 2 - 1 You need to solve, in the interval $0 < x < \pi$, the equation cos x = -1. $\pi x \pi O$ Draw the graph of $y = \cos x$. $\arccos(-1) = \pi c \pi 2 S \pi \pi$ You need to solve, in the interval - < x < ..., 22 the equation tan $x = \sqrt{3}$. π The angle to the horizontal is and, as tan is +ve, 3 it is in the 1st quadrant. A $\pi 3 T - \pi C 2 \pi \arctan(\sqrt{3}) = ..., 3$ You can verify these results using the sin-1, cos-1 and tan-1 functions on your calculator. Exercise 6E In this exercise, all angles are given in radians. 1 Without using a calculator, work out, giving your answer in terms of π : a arccos (0) b arcsin(1) c arctan(-1) 1 e arccos - ($\sqrt{2}$) 1 f arctan - ($\sqrt{3}$) π g arcsin(sin) 3 2 Find: a arcsin(2) + arcsin(-2) 1 P 1 1 1 2π h arcsin(sin) 3 c arctan(1) - arctan(-1) 3 Without using a calculator, work out the values of: a sin(arcsin (- 2)) c tan(arcsin(- 2)) 1 4 Without using a calculator, work out the exact values of: a sin(arccos(2)) 1 d sec(arctan(\sqrt{3})) 160 b cos(arcsin(- 2)) c tan(arcsin(- 2)) c tan(arcsin(tan(arccos(- e cosec(arcsin(-1)) f sin (2arcsin()) 2 1 2)) $\sqrt{2}$ $\sqrt{2}$ Trigonometric functions P E/P 5 Given that arcsin k = α , where 0 < k < 1 and α is in radians, write down, in terms of α , the first two positive values of x satisfying the equation sin x = k. π 6 Given that x satisfies arcsin x = k, where 0 < k < _____, 2 a state the range of possible values of x (1 mark) b express, in terms of x, i cos k ii tan k π Given, instead, that - __ < k < 0, 2 c how, if at all, are your answers to part b affected? P E/P 7 Sketch the graphs of: π a y = __ + 2 arcsin x 2 c y = arccos (2x + 1) (2 marks) b y = π - arctan x d y = -2 arcsin (-x) 8 The function f is defined as f : x \rightarrow arcsin x, -1 < x < 1, and the function g is such
that g(x) = f(2x). a Sketch the graph of y = f(x) and state the range of f. (3 marks) b Sketch the graph of y = g(x). (2 marks) c Define g in the form $g: x \mapsto \dots$ and give the domain of g. (3 marks) d Define E/P (4 marks) g-1 in the form $g: x \mapsto \dots$ and give the domain of g. (3 marks) d Define E/P (4 marks) g-1 in the form $g-1: x \mapsto \dots$ 9 a Prove that for 0 < x < 1, arccos $x = \arcsin \sqrt{1 - x^2}$ b Give a reason why this result is not true for -1 < x < 0. (2 marks) (4 marks) (2 equations to 1 decimal place. E/P 1 Solve tan $x = 2 \cot x$, in the interval $-180^\circ < x < 90^\circ$. (4 marks) E/P 2 Given that $p = 2 \sec \theta$ and $q = 4 \cot \theta$, show that p2q2 = 16(1 - p2). (4 marks) P 4 a Solve, in the interval $0 < \theta < 180^\circ$, i cosec $\theta = 2 \cot \theta$ ii 2 cot $2\theta = 7 \csc \theta - 8$ i $sec(2\theta - 15^\circ) = cosec 135^\circ$ ii $sec2 \theta + tan \theta = 3$ b Solve, in the interval $0 < \theta < 360^\circ$, c Solve, in the interval $0 < x < 2\pi$, $\pi 4$ i $cosec(x + __) = -\sqrt{2}$ ii $sec2 x = _3 15$ E/P P E/P 5 Given that 5 sin x cos y + 4 cos x sin y = 0, and that cot x = 2, find the value of cot y. 6 Prove that: a $(tan \theta + cot \theta)(sin \theta + cos \theta) = sec \theta + cosec \theta cosec x b_0$ $= \sec 2 \operatorname{x} \operatorname{cosec} \operatorname{x} - \sin \operatorname{x} \operatorname{c} (1 - \sin \operatorname{x})(1 + \operatorname{cosec} \operatorname{x}) = \cos \operatorname{x} \operatorname{cot} \operatorname{x} \operatorname{cos} \operatorname{x} \operatorname{d} \operatorname{x} - 1 + \sin \operatorname{x} 1 1 \operatorname{e} + 2 \operatorname{sec} \theta \tan \theta \operatorname{cosec} \theta - 1 \operatorname{cosec} \theta + 1 \operatorname{(sec} \theta - \tan \theta) \operatorname{(sec} \theta + \tan \theta) = \cos 2 \theta \operatorname{f} \operatorname{1} + \tan 2 \theta \sin \operatorname{x} 1 + \cos \operatorname{x} 7 \operatorname{a} \operatorname{Prove} \operatorname{that} + 2 \operatorname{cosec} \operatorname{x} \cdot 1 + \cos \operatorname{x} \operatorname{sin} \operatorname{x} \operatorname{sin} \operatorname{x} 1 + \cos \operatorname{x} \operatorname{sin} \operatorname{x} 1 + \cos \operatorname{x} \operatorname{x} \operatorname{a} \operatorname{cosec} \theta + 1 \operatorname{(sec} \theta - \tan \theta) \operatorname{(sec} \theta + \tan \theta) = \cos 2 \theta \operatorname{f} \operatorname{1} + \tan 2 \theta \sin \operatorname{x} 1 + \cos \operatorname{x} 7 \operatorname{a} \operatorname{Prove} \operatorname{that} + 2 \operatorname{cosec} \operatorname{x} \cdot 1 + \cos \operatorname{x} \operatorname{sin} \operatorname{sin$ $\equiv 2 \operatorname{cosec} x. 1 + \cos x \sin x (4)$ marks) π 9 Given that sec A = -3, where < A < π , 2 a calculate the exact value of tan A 162 (5 marks) c cot θ d cosec θ Trigonometric functions E π 11 Solve, in the interval 0 < x < 2 π , the equation sec(x +) = 2, giving your answers in 4 terms of π . (5 marks) E/P 12 Find, in terms of π , the value of arcsin(2) – arcsin(- 2). E/P 13 Solve, in the interval $0 < x < 2\pi$, the equation 1 1 (4 marks) sec2 x giving your answers in terms of π . E/P $= 2\sqrt{3}$ 3 tan x - 2 = 0, (5 marks) 14 a Factorise sec x cosec x - 2 sec x - cosec x + 2 = 0, in the interval $0 < x < 360^\circ$. E/P $= E/P \pi 15$ Given that $\arctan(x - 2) = -$, find the value of x. 3 (4 marks) (3 marks) 16 On the same set of axes sketch the graphs of y = cos x, 0 < x < \pi, and y = arccos x, -1 < x < 1, showing the coordinates of points at which the curves meet the axes. (4 marks) 17 a Given that sec x + tan x = -3, use the identity 1 + tan 2 x = sec 2 x to find the value of sec x - tan x. (3 marks) b Deduce the values of: i sec x ii tan x (3 marks) c Hence solve, in the interval $-180^\circ < x < 180^\circ$, sec x + tan x = -3. (3 marks) E/P 1 18 Given that p = sec θ - tan θ and q = sec θ + tan 2θ . (3 marks) b Hence solve, in the interval $-180^\circ < \theta < 180^\circ$, sec 4 θ = tan 4θ + 3 tan θ . (4 marks) π P 20 a Sketch

the graph of $y = \sin x$ and shade in the area representing $\int \sin x \, dx + \int \arcsin x \, dx = 2001 \text{ m}^2 \text{ F/P 21}$ Show that cot 60° sec 60° = $2\sqrt{3}$ 3 22 a Sketch, in the interval $-2\pi < 2\pi$, the graph of $y = 2 - 3 \sec x$. b Hence deduce the range of values of k for which the equation $2 - 3 \sec x = k$ has no solutions. P π 23 a Sketch the graph of $y = 3 \arctan (2 - 3 \sec x)$ (2 marks) (2 marks) (2 marks) (2 marks) (2 marks) (3 marks) (2 marks) (2 marks) (3 marks) (3 marks) (3 marks) (2 marks) (3 marks) (2 marks) (3 marks) 24 a Prove that for 0, x < 1, $\arccos x = \arctan \sqrt{1 - x^2}$ x b Prove that for -1 < x, 1, $\arccos x = k + \arctan \sqrt{1 - x^2}$ x, where k is a constant to be found. Summary of key points 1 1 • sec x = $\sin x 1 \cdot \cot x = 1$ tan $x \cos x \cdot \cot x = 1$ sin x (undefined for values of x for which $\cos x = 0$ (undefined for values of x for which $\sin x = 0$) (undefined for values of x for which $\tan x = 0$) 2 The graph of y = sec x, x $\in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$. y = sec x, y 1 -450° -270° -90° 90° 270° 450° 5π - 2 3π - 2 O - π 2 -1 π 2 3π $2 5\pi x 2 \cdot T$ he domain of y = sec x is x $\in \mathbb{R}$, x $\neq 90^\circ$, 270°, 450°, ... or any odd multiple of 90°. • In radians the domain is x $\in \mathbb{R}$, $\pi 3\pi 5\pi x \neq _$, ... or any odd multiple 2 2 2 π of 2 • The range of y = sec x is y < -1 or y > 1. 3 The graph of y = cosec x, x $\in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which sin x = 0. $y = cosec x y - 180^\circ - 2\pi - \pi O - 1164 \cdot In$ radians the domain is $x \in \mathbb{R}$, $x \neq 0$, $\pi, 2\pi$, ... or any multiple of $\pi 1 - 360^\circ 180^\circ$, $360^\circ \pi, 2\pi \cdot T$ he domain of y = cosec x is $x \in \mathbb{R}$, $x \neq 0$, $\pi, 2\pi$, ... or any multiple of $\pi 1 - 360^\circ 180^\circ$, $360^\circ \pi, 2\pi \cdot T$ he domain of y = cosec x is $x \in \mathbb{R}$, $x \neq 0$, $\pi, 2\pi$, ... or any multiple of $\pi 1 - 360^\circ 180^\circ$, $360^\circ \pi, 2\pi \cdot T$ he domain of y = cosec x is $x \in \mathbb{R}$, $x \neq 0$, $\pi, x \in \mathbb{R}$, has period 180°. or π radians. It has vertical asymptotes at all the values of x for which tan x = 0. $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0^{\circ}$, 180° , -360° , -180° , -360° , have no solutions for -1 < k < 1. 6 You can use the identity sin 2 x + cos 2 x = 1 to prove the following identities: • 1 + tan 2 x = sec 2 x • 1 + cot 2 x = cosec 2 x y \pi 2 7 The inverse function of sin x is called arcsin x. • The domain of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is $-1 < x < 1 \pi \pi$ • The range of y = arcsin x is -1 < x $1 \times O - 1 \pi - 2 \vee \pi 8$ The inverse function of cos x is called arccos x. • The domain of y = arccos x is -1 < x < 1 • The range of y = arccos x is -1 < x < 1 • The range of y = arccos x is -1 < x < 1 • The range of y = arccos x is -1 < x < 1 • The range of y = arccos x is -1 < x < 1 • The range of y = arccos x < 180° $\pi 2 - 19$ The inverse function of tan x is called arctan x. • The domain of y = arccos x is -1 < x < 1 • The range of y = arccos x < 180° $\pi 2 - 19$ The inverse function of tan x is called arctan x. $x < 90^{\circ} y = \arccos x 1 x O y \pi 2 y = \arctan x O x - \pi 2 165 7$ Trigonometry and modelling Objectives After completing this unit you should be able to: \bullet Prove and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-177 \bullet Solve trigonometry and modelling Objectives After completing this unit you should be able to: \bullet Prove and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand and use the double-angle formulae \rightarrow pages 167-173 \bullet Understand \rightarrow Pages 167-173 \bullet Understan 177-181 formulae \bullet Write expressions of the form a cos $\theta \pm b \sin \theta$ in the forms \rightarrow pages 181-186 R cos($\theta \pm \alpha$) or R sin($\theta \pm \alpha$) \bullet Prove trigonometric identities using a variety of identities \rightarrow pages 186-189 \bullet Use trigonometric functions to model real-life situations \rightarrow pages 189-191 Prior knowledge check 1 2 Find the exact values of: π a sin 45° b cos $_ 6 c 2 sin 2 x - sin x - 3 = 0 b cos(2x - 30^\circ) = _ 12 \leftarrow Year 1$, Chapter 10 Prove the following: a cos x + sin x tan x = sec x + c ____ = sin 2 x 1 + cot 2 x cos 2 x 166 \leftarrow Section 5.4 Solve the following equations in the interval 0 < x, 360° . a sin(x + 50°) = $-0.9 3 \pi c tan _ 3 sin 2 x b cot x sec x sin x = 1 \leftarrow Section 6.4$ The strength of microwaves at different points within a microwave oven can be modelled using trigonometric \rightarrow Exercise 7G Q7 functions. Trigonometry and modelling 7.1 Addition formulae for sine, cosine and tangent are defined as follows: Notation The addition formulae are sometimes called the compound-angle formulae. $\blacksquare sin(A + B) \equiv sin A cos B + cos$ A sin B sin(A - B) = sin A cos B - cos A sin B \blacksquare cos(A + B) = cos A cos B - sin A sin B cos(A - B) = cos A cos B + sin A sin B tan A + tan B \blacksquare tan(A + B) = 1 + tan A tan B tan A - tan B tan(A - B) = 1 + tan A tan B tan A - tan B tan(A - B) = 1 + tan A tan B tan A - tan B tan A $\angle CAE = \beta$ and AE = 1. Additionally, lines AB and BC are perpendicular, lines AB and DE are perpendicular, lines AC and EC are perpendicular, lines AC and EC are perpendicular. 1 F Use the diagram, together with known properties of sine and cosine, to prove the following identities: a sin($\alpha + \beta$) = sin $\alpha \cos \beta + \cos \alpha \sin \beta$ b cos($\alpha + \beta$) = cos α $\cos \beta - \sin \alpha \sin \beta$ The diagram can be labelled with the following lengths using the properties of sine and cosine. $\cos (\alpha + \beta) \in \cos \alpha \sin \beta \sin \alpha \cos \beta = 0$ So AC = $\cos \beta = 0$ So AC = In triangle ACE, $\sin \beta = -3 \sin \beta = -1$ AE So EC = $\sin \beta$. FE FE In triangle FEC, $\cos \alpha = -3 \cos \alpha = -3 \cos \alpha = -3 \cos \alpha = -3 \cos \alpha = -3 \sin \alpha$ AC cos β So AB = cos α cos β . 167 Chapter 7 a Using triangle ADE DE = sin ($\alpha + \beta$) AD = cos $\alpha < \beta + cos \alpha < \beta - sin \alpha < \alpha < \beta - sin \alpha$ \overline{ADE} with angle ($\alpha + \beta$). You can see these
relationships more easily on the diagram by looking at AG = DE and GE = AD. Substitute the lengths from the diagram. Online Explore the proof step-by-step using GeoGebra. Example 2 Use the results from Example 1 to show that a cos (A - B) = cos A cos B + sin A sin B tan A + tan B b tan (A + B) = $1 - \tan A \tan B = \operatorname{Replace} B \text{ by } -B \text{ in } \cos (A + B) \equiv \cos A \cos B - \sin A \sin B \cos (A + (-B)) \equiv \cos A \cos (-B) - \sin A \sin (-B) \equiv \cos A \cos B + \sin A \sin B \cos (-B) = -\sin B \leftarrow \operatorname{Year} 1$, Chapter 9 sin (A + B) b tan (A + B) = $\cos (A + B) \sin A \cos B + \cos A \sin B \equiv$ - sin A sin B Divide the numerator and denominator by cos A cos B. sin A cos B _____ cos A sin B + _____ $\cos A \cos B \cos A \cos B \tan A + \tan B \equiv$ cos A cos B sin A sin B – as required $1 - \tan A \tan B$ Example 3 Prove that $\cos (A + B) \cos A$ $\cos A \cos B \cos A \cos B \equiv$ $- \equiv \cos B \sin B \sin B \cos B 168$ Cancel where possible. Trigonometry and modelling $\cos A \sin A LHS \equiv _$ _____ sin B $\cos B$ Write both fractions with a common denominator. $\cos A \cos B \sin A \sin B \equiv _$ _____ - ____ $\sin B \cos B \sin B \cos B \cos A \cos B - \sin A \sin B \equiv$ $\sin B \cos B \cos (A + B) \equiv$ \equiv RHS sin B cos B Example Problem-solving When proving an identity, always keep an eye on the final answer. This can act as a guide as to what to do next. Use the addition formula in reverse: cos A cos B - sin A sin B \equiv cos (A + B) 4 Given that 2 sin (x + y) = 3 cos (x - y), express tan x in terms of tan y. Expanding sin (x + y) and cos (x - y $2 - 3 \tan y$ So Exercise sin x Remember tan x = $\cos x \cos y$ will produce tan x and tan y terms. Collect all tan x terms on one side of the equation. Factorise. 7A 1 In the diagram $\angle BAC = \beta$, $\angle CAF = \alpha - \beta$ and AC = 1. Additionally lines AB and BC are perpendicular. C F a Show each of the following: $i \angle FAB = \alpha$ ii $\angle ABL$ $= \alpha$ and $\angle ECB = \alpha$ iii AB = cos β iv BC = sin β 1 b Use $\triangle BBC$ to write an expression for the lengths i CE ii BE d Use $\triangle BEC$ to write an expression for the lengths i CE ii BE d Use $\triangle BEC$ to write an expression for the lengths i CE ii BE d Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i CE ii BE d Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ to write an expression for the lengths i AD ii BD B α - β c Use $\triangle BEC$ c Use $\triangle BEC$ c $\cos \beta + \sin \alpha \sin \beta$ 169 Chapter 7 P 2 Use the formulae for $\sin (A - B)$ and $\cos (A - B)$ to show that $\tan A - \tan B$ tan $(A - B) \equiv 1 + \tan A$ tan B = -Q into the addition formula for $\sin (A - B) \equiv \sin P \cos Q - \cos P \sin Q$. P 4 A student makes the mistake of thinking that $\sin (A - B) \equiv \sin P \cos Q - \cos P \sin Q$. P 4 A student makes the mistake of thinking that $\sin (A - B) \equiv \sin P \cos Q - \cos P \sin Q$. $A + \sin B$. Choose non-zero values of A and B to show that this identity is not true. P P Watch out This is a common mistake. One counter-example is sufficient to disprove the statement. 5 Using the expansion of $\cos (A - B)$ with $A = B = \theta$, show that $\sin (2 - \theta) = \cos \theta$. $2 \pi b$ Use the expansion of $\cos (A - B)$ to show that $\cos (_ - \theta) = \sin \theta$. 2 P π 7 Write $\sin (x + _)$ in the form p sin x + q cos x where p and q are constants to be found. 3 P 9 Express the following as a single sine, cosine or tangent: a sin 15° cos 20° + cos 15° sin 20°) cos () - sin () sin () 2 2 2 2 P 10 Use the addition formulae for sine or cosine to write each of the following as a single π trigonometric function in the form sin (x ± θ), where $0 < \theta < 21$ (sin x + cos x) a $\sqrt{2}$ 170 1 b (cos x - sin x) $\sqrt{2}$ 1 c (sin x + $\sqrt{3}$ cos x) 2 1 (sin x - cos x) d – 2v j cos ($\sqrt{2}$ Trigonometry and modelling P 11 Given that $\cos y = \sin (x + y)$, show that $\tan y = \sec x - \tan x$. P 12 Given that $\sin x (\cos y + 2 \sin y) = \cos x (2 \cos y - \sin y)$, find the value of $\tan (x + y)$. P 14 In each of the following, calculate the exact value of $\tan x$. a $\tan (x - 45^\circ) = 4$ 1 Hint First multiply out the brackets. b sin $(x - 60^\circ) = 3 \cos(x + 30^\circ) E/P_{\pi} 1 15$ Given that $\tan(x + _) = _$, show that $\tan x = 8 - 5\sqrt{3} \cdot 32 E/P 16$ Prove that $c \tan(x - 60^\circ) = 2$ (3 marks) You must show each stage of your working. Challenge This triangle is constructed from two right-angled triangles T1 and T2. a Find expressions involving x, y, A and B for: i the area of T1 ii the area of T1 ii the area of T2 iii the area of T1 ii the area of T2 iii th the angle addition formulae The addition formulae can be used to find exact values of trigonometric functions of different angles. Example 5 Show, using the formula for sin (A - B), that sin 15° = sin 45° cos 30° - cos 45° sin 30° _ _ _ 1 $\sqrt{1} \sqrt{1} \sqrt{1} \sqrt{2}$ ($\sqrt{2} \sqrt{2}$) - (22)(23) 4 You know the exact values of sin and cos for many angles, e.g. 30° , 45° , 60° , 90° , 180° ..., so write 15° using two of these angles. You could also use sin ($60^\circ - 45^\circ$). $\sqrt{6} - \sqrt{2}$ 4 171 Chapter 7 Example 6 Given that sin A = -5 and $180^\circ < A < 270^\circ$, and that cos B = -13 and B is obtuse, find the value of: 3 a cos (A - 4171 Chapter 7 Example 6 Given that sin A = -5 and $180^\circ < A < 270^\circ$, and that cos B = -13 and B is obtuse, find the value of: 3 a cos (A - 4171 Chapter 7 Example 6 Given that sin A = -5 and $180^\circ < A < 270^\circ$, and that cos B = -13 and B is obtuse, find the value of: 3 a cos (A - 4171 Chapter 7 Example 6 Given that sin A = -5 and $180^\circ < A < 270^\circ$. B) 12 b tan (A + B) c cosec (A - B) You know sin A and cos B, but need to find sin B and cos A. a cos $(A - B) \equiv \cos A \cos B + \sin A \sin B \cos 2 A \equiv 1 - \cos 2 = 1 - (-5)$ 12 = 1 - (-5) 12
= 1 - (-5) 12 = 1 $3 12 \cos (A - B) = (-5)(-13) + (-5)(+13) 48 15$ $33 = 65 - 65 = 65 b \tan (A + B) = Use sin 2x + cos 2x = 1 to determine which one to use. cos x is negative in the$ obtuse so sin B = 1345third quadrant, so choose the negative square root -45. sin x is positive in the second quadrant (obtuse angle) so choose the positive square root. tan A + tan B 1 - tan A tan B 5 34 + (-12) So tan (A + B) = 531 - (4)(-12) 134816 $1 = 3 \times 63 = 6363$ Substitute the values for sin A, sin ____ (65) 13 1 Remember cosec x = ____ sin x Exercise 7B 1 Without using your calculator, find the exact value of: a cos 15° 172 b sin 75° c sin (120° + 45°) d tan 165° Trigonometry and modelling 2 Without using your calculator, find the exact value of: a sin 30° cos 60° + cos 30° sin 60° cos 15° - cos 60° sin 15° b cos 110° cos _____1 + tan 15° c sin 33° cos 27° + cos 33° sin 27° 7π π tan ____ – tan ____3 12 i _____ $20^{\circ} + \sin 110^{\circ} \sin 20^{\circ} \pi \pi \pi \pi \pi d \cos \cos - \sin \sin 88886 \cos 70^{\circ} (\cos 50^{\circ} - \tan 70^{\circ} \sin 50^{\circ}) \tan 45^{\circ} + \tan 15^{\circ} g$ 1 - tan 45° tan 15° 1 - tan 15° h _ 7пп ___ 1 + tan tan __ 3 12 Е __ j √ 3 cos 15° – sin 15° 3 a Express tan (45° + $\sqrt{2} - \sqrt{6}$ 4 (4 marks) b Hence, or otherwise, show that sec $105^\circ = -\sqrt{a} (1 + \sqrt{2})^2$ 30°) in terms of tan 45° and tan 30°. (2 marks) b Hence show that tan 75° = $2 + \sqrt{3}$. P E/P (2 marks) 4 Given that cot A = 4 and cot (A + B) = 2, find the value of cot B. 1 5 a Using cos ($\theta + \alpha$) = cos θ cos α - sin θ sin α , or otherwise, show that cos 105° = \sqrt{b}), where a and b are constants to be found. (3 marks) P 6 Given that sin A = 5 and sin B = 2, where A and B are both acute angles, calculate the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A + B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A - B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A - B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A - B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle measured in radians, find the exact value of: $\pi \pi d \tan(A - B) P P b \cos(A - B)$ 7 Given that cos A = -5, and A is an obtuse angle meas 3 4 8 Given that sin A = 17, where A is acute, and cos B = -5, where B is obtuse, calculate the exact value of: 8 a sin (A - B) P 4 b cos (A - B) - 24, where A is reflex, and sin B = 13, where A is reflex, and sin B = 13, where A is reflex, and sin B = 13, where B is obtuse, calculate the exact value of: 8 a sin (A - B) P 4 b cos (A - B) - 24, where A is reflex, and sin B = 13, where A is reflex, and sin B = 13, where B is obtuse, calculate the exact value of: 7 a sin (A - B) P c sec (A - B) - 24. $\tan A = 5$ and $\tan B = 3$, calculate, without using your calculator, the value of A + B in degrees, where: 1 2 a A and B are both acute, b A is reflex and B is acute. 173 Chapter 7 7.3 Double-angle formulae to derive the following double-angle formulae. a $\cos 250^\circ - \sin 250^\circ c$ π $\sec 70^\circ 1 - \tan 26 a \cos 250^\circ - \sin 250^\circ = \cos (2 \times 50^\circ) = \cos 100^\circ \pi 2 \tan 6 \pi$ 1 – tan2 A Example 7 Use the double-angle formulae to write each of the following as a single trigonometric ratio. π 2 tan ___ 4 sin 70° 6 b _____ in reverse, with $A = 61 - \tan 2 A 2\pi \pi \tan = \tan 634 \sin 70^\circ c$ = $4 \sin 70^\circ \cos 70^\circ \sec 70^\circ = 2(2 \sin 70^\circ \cos 70^\circ) = 2 \sin 140^\circ \text{Example } 11$ sec $x = \cos x \operatorname{so} \cos x = \sec x \operatorname{Recognise this}$ is a multiple of $2 \sin A$ $-\tan 26$ Use $\cos 2A \equiv \cos 2A - \sin 2A$ in reverse, with $A = 50^{\circ}$. $\pi 2 \tan A$ Use $\tan 2A \equiv$ $\cos A$. Use $\sin 2A \equiv 2 \sin A \cos A$ in reverse with $A = 70^{\circ}$. 8 Given that $x = 3 \sin \theta$ and $y = 3 - 4 \cos 2\theta$, eliminate θ and express y in terms of x. The equations can be written as $x \sin \theta = -33 - y \cos 2\theta = -4$ Watch out Be careful with this manipulation. Many errors can occur in the early part of a solution. As $\cos 2\theta \equiv 1 - 2 \sin 2\theta$ for all values of 4 So or $174 \times 2 = 1 - 2($) 3θ has been eliminated from this equation. We still need to solve for y. y $\times 21 = 2($) $- 443 \times 2y = 8($) - 13 The final answer should be in the form y = ... Trigonometry and modelling Example 9 Given that $\cos x = 4$, and that $180^\circ < x < 360^\circ$, find the exact value of: $3 = 3 \times 2y = 8($) - 13 The final answer should be in the form y = ... Trigonometry and modelling Example 9 Given that $\cos x = 4$, and that $180^\circ < x < 360^\circ$, find the exact value of: $3 = 3 \times 2y = 8($ $\cos 2 A$ Use $\sin 2 A + \cos 2 A = 1$ to determine $\sin A = 1 - (4) 327 = 16 180^\circ < A < 360^\circ$, so $\sin A = -\sin 2x = 2 \sin x \cos x \sqrt{7} 4 \sin A$ is negative in the third and fourth quadrants, so choose the negative square root. $= 2(-)() = -448\sqrt{7} 3 \sqrt{7} - \sqrt{7} 4 \sin x$ b tan $x = \cos x = 3$ Find tan x in simplified surd form, then substitute this value into the double-angle formula for tan 2x. 4 = $-\sqrt{7}$ 3 2 tan x ______ tan 2x = 71 - tan 2 x 1 - Make sure you square all of tan x when working out tan 2 x: 9 2 = $-2\sqrt{7}$ 3 × 9 7 (3) = $9\sqrt{7}$ - 2 = $-3\sqrt{7}$ Exercise 7C P 1 Use the expansion of sin (A + B) to show that $\sin 2A \equiv 2 \sin A \cos A$. $P 2 a Using the identity <math>\cos (A + B) \equiv \cos A \cos B - \sin A \sin B$, show that $\cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A \equiv 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \cos 2A =
1 - 2 \sin 2 A - 1 ii \cos 2A = 1 - 2 \sin 2 A - 1 ii \sin 2 A$ Write each of the following expressions as a single trigonometric ratio. a 2 sin 10° cos 10° b 1 – 2 sin 2 25° c cos 2 40° – sin 2 40° 2 tan 5° d _____ 1 – tan 2 5° 1 e ____ $2 \sin (24.5)^\circ \cos (24.5)^\circ f 6 \cos 2 30^\circ - 3 \sin 8^\circ g$ sec $8^\circ \pi \pi h \cos 2 - \sin 2$ 16 16 5 Without using your calculator find the exact values of: a 2 sin 22.5° cos $22.5^{\circ} b 2 \cos 2 15^{\circ} - 1 c (\sin 75^{\circ} - \cos 75^{\circ}) 2 \pi 2 \tan 8 d$ $\pi 1 - \tan 2 b \text{ Hence find the exact value of } (\sin + \cos A) 2 \equiv 1 + \sin 2A$. $\pi \pi 2 b \text{ Hence find the exact value of } (\sin + \cos A) 2 \equiv 1 + \sin 2A$. $\pi \pi 2 b \text{ Hence find the exact value of } (\sin + \cos A) 2 \equiv 1 + \sin 2A$. $\pi \pi 2 b \text{ Hence find the exact value of } (\sin A + \cos A) 2 \equiv 1 + \sin 2A$. $a \cos 2 3\theta - \sin 2\theta$ $3\theta \theta 1 - \tan 2 _ 2 _ \theta f \sin 2\theta \cos 2\theta e \sqrt{1 + \cos 2\theta} d 2 - 4 \sin 2 _ 2 \tan \theta g 4 \sin \theta \cos \theta \cos 2\theta h _ i \sin 4\theta - 2 \sin 2\theta \cos 2\theta + \cos 4\theta \sec 2\theta - 2P 8$ Given that $p = 2 \cos \theta$ and $q = \cos 2\theta$, express q in terms of p. P 9 Eliminate θ from the following pairs of equations: a $x = \cos 2\theta$, $y = 1 - \cos 2\theta b x = \tan \theta$, $y = \cot 2\theta c x = \sin \theta$, $y = \sin \theta$, $y = \sin \theta x = \cos 2\theta + 1$, $y = 2 \sin \theta P 10$ Given that $\cos x = _4$, find the exact value of $\cos 2\theta x$. P 11 Find the possible values of $\sin \theta$ when $\cos 2\theta = _25P 12$ Given that $\tan \theta = _4$, and that θ is acute, 1 23 3 a find the exact value of $\sin 4\theta$. 176 i tan 2 θ ii $\sin 2\theta$ iii $\cos 2\theta$ Trigonometry and modelling P 13 Given that $\cos A = _3$, and that A is obtuse, 1 a find the exact value of: i cos 2A ii sin A iii cosec 2A $4\sqrt{2}$ b show that tan 2A = 7 E/P $3\pi \theta 3 14$ Given that cos x + sin x = m and cos x - sin x = n, where m and n are constants, write down, in terms of m and n, the value of cos 2x. (4) marks) E/P 16 In \triangle PQR, PQ = 3 cm, PR = 6 cm, QR = 5 cm and \angle QPR = 20. a Use the cosine rule to show that cos 20 = 9 (3 marks) b Hence find the exact value of sin 0. (2 marks) b Hence find the scales on each axis are the same, and that 1 makes an angle θ with the x-axis, 3 a write down the value of tan θ b show that m = E/P(1 marks) 7 18 a Use the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, to show that $\cos 2A \equiv 2 \cos 2A - 1$. (2 marks) The curves C1 and C2 have equations C1: $y = 4 \cos 2x C2$: $y = 6 \cos 2x - 3 \sin 2x$ b Show that the x-coordinates of the points where C1 and C2 intersect satisfy the equation (3 marks) $\cos 2x + 3 \sin 2x - 3 = 0$ P sin 2A 19 Use the fact that $\tan 2A \equiv$ to derive the cos 2A formula for tan 2A in terms of tan A. Hint Use the identities for sin 2A and $\cos 2A$ and then divide both the numerator and denominator by $\cos 2A$. 7.4 Solving trigonometric equations You can use the addition formulae and the double-angle formulae to help you solve trigonometric equations. 177 Chapter 7 Example 10 Solve $4 \cos(\theta - 30^\circ) = 8\sqrt{2} \sin \theta$ in the range $0 < \theta < 360^\circ$. Round your answer to 1 decimal place. $4 \cos(\theta - 30^\circ) = 8\sqrt{2} \sin \theta$ is $\theta \sin 30^\circ = 8\sqrt{2} \sin \theta$ Use the formula for $\cos(A - B)$. $\theta(-) + 4 \sin \theta(-) = 8\sqrt{2} \sin \theta + 2 \sin \theta$ 200.4° Use a CAST diagram or a sketch graph to find all the solutions in the given range. Example 11 Solve $3\cos 2x - \cos x + 2 = 0$ for $0 < x < 360^{\circ}$. Using a double angle formula for $\cos 2x 3\cos 2x - \cos x + 2 = 6\cos 2x - 3 - \cos x + 2 = 6\cos 2x$ $\cos x = -3$ or $0\ 0\ 0\ \cos x = 2\ y = \cos x\ y = 1\ 2\ 90^\circ\ 1\ \cos -1(-3) = 109.5^\circ\ 180^\circ\ 360^\circ\ x\ y = -1\ 3\ 270^\circ\ 1\ \cos -1(-3) = 60^\circ\ 50\ x = 60^\circ\ 109.5^\circ\ 250.5^\circ\ 300^\circ\ 178$ Choose the double angle formula for $\cos 2x\ which\ only\ involves\ \cos x = 2\ y = \cos 2x\ - 1\ This\ will\ give\ you\ a\ quadratic\ equation\ in\ \cos x$. This quadratic equation factorises: $6X\ 2 - X - 1$ = (3X + 1)(2X - 1) 1 v O Problem-solving Trigonometry and modelling Example 12 Solve 2 tan 2y tan y = 3 for $0 < y < 2\pi$. Give your answers to 2 decimal places. 2 tan 2y tan y = 3 2 tan y 2 tan y = 3 2 tan y 2 tan y = 3 (1 - tan 2y) 4 tan y = 2 (1 - tan 2y) 4 tan y = 3 (1 - tan 2y) 4 tan y = $\tan 2y = 3 - 3 \tan 2y = 2 - 3 \tan 2y = 3 - 3 \tan 2y = 2 - 3 \tan 2$ $16 \sin 3\theta - 12 \sin \theta - 2\sqrt{3} = 0 \text{ giving your answers in terms of } \pi. a LHS = \sin 3A = \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A = 2 \sin A \cos A + \cos 2A \sin A = 2 \sin A \cos A + \cos 2A \sin A = 2 \sin A
\cos A + \cos 2A \sin A = 2 \sin A \cos A + \cos 2A \sin A = 2 \sin A - 2 \sin 3A = 2 \sin A - 2 \sin 3A = 2 \sin A - 2 \sin 3A = 3 \sin A - 4 \sin 3A = RHS b 16 \sin 3\theta - 12 \sin \theta - 2\sqrt{3} = 0$ best identity to use. Use sin2 A + cos2 A = 1 to substitute for cos2 A. Problem-solving The question says 'hence' so look for an opportunity to use the identity by -4. Use a CAST diagram or a sketch graph to find all answers for $3\theta < \theta < 2\pi$ so $0 < 3\theta < 6\pi$. 179 Chapter 7 Exercise P 7D 1 Solve, in the interval $0 < \theta$, 360°, the following equations. Give your answers to 1 d.p. a 3 cos $\theta = 2 \sin(\theta + 60^\circ) c \cos(\theta + 25^\circ) + \sin(\theta + 65^\circ) = 1 E/P b \sin(\theta + 30^\circ) + 2 \sin \theta = 0 d \cos \theta = \cos(\theta + 60^\circ) \pi 1$ 2 a Show that $\sin(\theta + 1) = (\sin \theta + \cos \theta) 4\sqrt{2} 1$ b Hence, or otherwise, solve the equation $(\sin \theta + \cos \theta) = 1 C/P b \sin(\theta + 65^\circ) = 1 E/P b \sin(\theta + 6$ 2π . $\sqrt{2}\sqrt{2}$ (2 marks) (4 marks) c Use your answer to part b to write down the solutions to $\sin \theta + \cos \theta = 1$ over the same interval. (2 marks) P 3 a Solve the equation $\cos \theta \cos 30^\circ - \sin \theta \sin 30^\circ = 0.5$, for $0 < \theta < 360^\circ$. b Hence write down, in the same interval, the solutions of $\sqrt{3} \cos \theta - \sin \theta = 1$. P 4 a Given that $3 \sin (x - y) - \sin (x + y) = 0$, show that tan x = 2 tan y. b Solve 3 sin (x - 45°) - sin (x + 45°) = 0, for 0 < x < 360°. P 5 Solve the following equations, in the intervals given. a sin $2\theta = \sin \theta$, $0 < \theta < \pi c 3 \cos 2\theta = 1 - \cos \theta$, -1 = 0, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < 360°$ g 2 sin $\theta = \sec \theta$, $0 < \theta < \pi c 3 \cos 2\theta = 1 - \cos \theta$, -1 = 0, $0 < \theta < \pi c 3 \cos 2\theta = 1 - \cos \theta$, -1 = 0, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c 3 \cos 2\theta = 2 \cos 2\theta$, $0 < \theta < \pi c$ 2π i 2 tan $\theta = \sqrt{3}$ (1 - tan θ)(1 + tan θ), $0 < \theta < 2\pi$ k 4 tan $\theta = \tan 2\theta$, $0 < \theta < 360^\circ$ d sin $4\theta = \cos 2\theta$, $0 < \theta < \pi$ h 2 sin $2\theta = 3$ tan θ , $0 < \theta < \pi$ h 2 sin $2\theta = 2 \sin 2\theta$, -180° j sin $2\theta = 2 \sin 2\theta$, -180° sin $5 \sin 2\theta + 4 \sin \theta = 0$ can be written in the form a sin θ (b cos $\theta + c$) = 0, stating the values of a, b and c. E/P b Hence solve, for $0 < \theta$, 360° , the equation $5 \sin 2\theta + 4 \sin \theta = 0$. (4 marks) $\theta = 0$. (5 marks) $\theta = 0$. (7 marks) $\theta = 0$. (8 marks) $\theta = 0$. (9 marks) θ solve the equation $\sin 2\theta + \cos 2\theta = 1$ for $0 < \theta$, 360° . E/P (2 marks) 1 b Use the result to solve, for $0 < \theta$, π , the equation $\cos 2\theta - \sin 2\theta = \sqrt{2}$ Give your answers in terms of π . 180 (4 marks) (3 marks) Trigonometry and modelling P 10 a Show that: θ 2 tan 2 i sin $\theta = \theta$ 1 + tan 2 2 θ 1 - tan 2 2 ii cos $\theta = \sqrt{2}$ Give your answers in terms of π . 180 (4 marks) (3 marks) Trigonometry and modelling P 10 a Show that: θ 2 tan 2 i sin $\theta = \theta$ 1 + tan 2 2 θ 1 - tan 2 2 ii cos $\theta = \sqrt{2}$ Give your answers in terms of π . θ b By writing the following equations as quadratics in tan , solve, in the interval $0 < \theta < 360^\circ$: 2 i sin $\theta + 2 \cos \theta = 1$ ii $3 \cos \theta - 4 \sin \theta = 2 \overline{E/P} 11$ a Show that $3 \cos 2x - \sin 2x \equiv 1 + 2 \cos 2x$. (3 marks) b Hence sketch, for $-\pi < x < \pi$, the graph of y = coordinates of points where the curve meets the axes. $3 \cos 2x E/P E/P - \sin 2x$, showing the θ θ 12 a Express 2 cos2 $-4 \sin 2$ in the form a cos θ + b, where a and b are constants. 2 2 θ θ b Hence solve 2 cos2 $-4 \sin 2$ = -3, in the interval $0 < \theta$, 360°. 2 2 13 a Use the identity sin 2 A + cos 2 A = 1 to show that sin 4 A + cos 4 A = 2 (2 - sin 2 2A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 2A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 2A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 -
sin 2 2A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b Deduce that sin 4 A + cos 4 A = 2 (2 - sin 2 A). 1 b 4A). c Hence solve $8 \sin \theta + 8 \cos \theta = 7$, for 0, θ , π . (3 marks) Hint E/P (3 marks) Biart by squaring (sin $2 A + \cos 2 A$). 14 a By writing 3θ as $2\theta + \theta$, show that $\cos 3\theta = 4 \cos 3\theta - 3 \cos \theta$. (4 marks) b Hence, or otherwise, for 0, θ , π , solve $6 \cos \theta - 8 \cos 3\theta + 1 = 0$ giving your answer in terms of π . (5 marks) 7.5 Simplifying a cos $x \pm b \sin x$ Expressions of the form a $\cos x + b \sin x$, where a and b are constants can be written as a sine function only or a cosine function only or a cosine function only. \blacksquare You can write any expression of the form a $\cos \theta + b \sin \theta$ as either R sin $(x \pm \alpha)$ where R > 0 and $0 < \alpha < 90^\circ$, or \blacksquare R $\cos \alpha = a$, R $\sin \alpha = b$ and R $\sqrt{a^2 + b^2}$ Use the addition formulae to expand sin $(x \pm \alpha)$ or cos $(x \pm \beta)$, then equate coefficients. 181 Chapter 7 Example 14 Show that you can express 3 sin $x + 4 \cos x$ in the form: a R sin $(x \pm \alpha)$ b R cos $(x - \alpha)$ where R > 0, $0 < \alpha < 90^\circ$, $0 < \beta < 90^\circ$ giving your values of R, α and β to 1 decimal place when appropriate. a R sin $(x \pm \alpha)$ b R cos $(x \pm \alpha)$ b R c $R2 \cos 2 \alpha + R2 \sin 2 \alpha = 32 + 42 \text{ Square and add the equations to eliminate } \alpha \text{ and find } R2. R2 = 25, \text{ so } R = 5 \text{ Use } \sin 2 \alpha + \cos 2 \alpha \equiv 1. R2 (\cos 2 \alpha + \sin 2 \alpha) = 25 3 \sin x + 4 \cos x \equiv R \cos x \cos \beta + R \sin x \sin \beta \text{ So } R \cos \beta = 4 \text{ and } R \sin \beta = 3 R \sin \beta = 3 R \sin \beta$ $R \cos \beta 3 =$ $\tan \beta = 4$ Use $\cos (A - B) \equiv \cos A \cos B + \sin A \sin B$ and multiply through by R. Equate the coefficients of the $\cos x$ and $\sin x$ terms. Divide the equations to eliminate R. So $\beta = 36.9^{\circ}$ (1 d.p.) R2 $\cos 2\beta + R2 \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R2 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. R2 ($\cos 2\beta + \sin 2\beta = 32 + 42$ Square and add the equations to eliminate α and find R2. R3 = 25, so R = 5 Remember sin $\alpha + \cos 2\alpha \equiv 1$. β = 25 3 sin x + 4 cos x = 5 cos (x - 36.9°) y Online Explore how you can transform the graphs of y = sin x and y = cos x to obtain the graph of y = 3 sin x + 4 cos x using technology. 182 x Trigonometry and modelling Example 15 $_{-}$ π a Show that you can express sin x - $\sqrt{3}$ cos x in the form R sin (x - α), where R > 0, 0 < α < $_{-}$ 2 $_{-}$ b Hence sketch the graph of $y = \sin x - \sqrt{3} \cos x$. a Set $\sin x - \sqrt{3} \cos x \equiv R \sin (x - \alpha) \sin x - \sqrt{3} \cos x \equiv R \sin x \cos \alpha - R \cos x \sin \alpha$. So $R \cos \alpha = 1$ and $R \sin \alpha = \sqrt{3}$, so $\alpha = 2 \sin (x - \sqrt{3} \cos x \equiv 2 \sin x = 2 \sin x))$ Equate the coefficients of sin x and cos x on both sides of the identity. y 2 - 3π - π O 2 2 π 4π 3π π π 32 3 2 2π 7π x 3 π You can sketch y = 2 sin (x - _) by 3 π _ translating y = sin x by to the right and 3 then stretching by a scale factor of 2 in the y-direction. -2 Example 16 a Express 2 cos θ + 5 sin θ in the form R cos (θ - α), where R > 0, 0 < α < 90°. b Hence solve, for $0 < \theta < 360^\circ$, the equation $2\cos\theta + 5\sin\theta = 3$. a Set $2\cos\theta + 5\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$ So $R\cos\alpha = 2$ and $R\sin\alpha = 55$ Dividing $\tan \alpha = 1$, so $\alpha = 68.2^\circ 2$ Squaring and adding: $R = \sqrt{29}$ So $2\cos\theta + 5\sin\theta = \sqrt{29}\cos(\theta - 68.2^\circ) = 33$ So $\cos(\theta - 68.2^\circ) = 33$ So $\cos(\theta - 68.2^\circ) = 100$ $(\sqrt{29}) = 56.1...^{\circ}$ So $\theta - 68.2^{\circ} = -56.1...^{\circ}$, $56.1...^{\circ} \theta = 12.1^{\circ}$, 124.3° (to the nearest 0.1°) Equate the coefficients of sin x and cos x on both sides of the identity. Use the result from Divide both sides by $\sqrt{29}$. As $0 < \theta < 360^{\circ}$, the interval for $(\theta - 68.2^{\circ})$ is $-68.2^{\circ} < \theta - 68.2^{\circ} < 29\overline{1.8}^{\circ}$. is positive, so solutions for $\theta - 68.2^\circ$ are in $\sqrt{29}$ the 1st and 4th quadrants. 183 Chapter 7 Example 17 $f(\theta) = 12 \cos \theta + 5 \sin \theta = R \cos (\theta - \alpha)$. b Find the maximum value of $f(\theta)$ and the smallest positive value of θ at which it occurs. a Set $12 \cos \theta + 5 \sin \theta = R \cos \theta \cos \alpha + R \sin \theta$ $\overline{\sin \alpha \text{ So } R \cos \alpha} = 12$ and $R \sin \alpha = 5$ Online x Use technology to explore maximums and minimums of curves in the form R cos ($\theta - \alpha$). Equate sin x and cos x terms and then solve for R and α . 5 R = 13 and tan $\alpha = 12 \Rightarrow \alpha = 22.6^{\circ}$ So 12 cos $\theta + 5 \sin \theta \equiv 13 \cos (\theta - 22.6^{\circ})$ b The maximum value of 13 cos ($\theta - 22.6^{\circ}$) is 13. This occurs when cos ($\theta - \alpha$). $22.6^\circ) = 1 \theta - 22.6^\circ = \dots, -360^\circ, 0^\circ, 360^\circ, \dots$ The smallest positive value of θ is 22.6° . Exercise The maximum value of cos x is 1 so the maximum value of cos x is 1 so the maximum value of θ . 7E Unless otherwise stated, give all angles to 1 decimal place and write non-integer values of R in surd form. 1 Given that $5 \sin \theta + 12 \cos \theta \equiv R \sin (\theta + \alpha)$, find the value of R, R > 0, and the value of $\tan \alpha$. 2 Given that $\sqrt{3} \sin \theta + \sqrt{6} \cos \theta \equiv 3 \cos (\theta - \alpha)$, where $0 < \alpha < 90^\circ$, find the value of α . $\pi 4$ a Show that $\cos \theta - \sqrt{3} \sin \theta$ can be written in the form $R \cos (\theta - \alpha)$. $-\alpha$), with R > 0 and 0 < α < 2 b Hence sketch the graph of y = $\cos \theta - \sqrt{3} \sin \theta$, 0 < α < 2π , giving the coordinates of P. c Write down the maximum and minimum values of 7 cos θ - 24 sin θ = 15 i 7 cos θ - 24 sin θ = 15 i 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 15 i 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26 iii 7 cos θ - 24 sin θ = 26
iii 7 cos θ - 24 sin θ = 26 iii 7 cos Hence, or otherwise, solve f(x) = 2 for $0 < \theta$, 360° . 184 (4 marks) (3 marks) Trigonometry and modelling E π 7 a Express cos $2\theta - 2 \sin 2\theta$ in the form R cos ($2\theta + \alpha$), where R > 0 and $0 < \alpha < 2$ Give the value of α to 3 decimal places. b Hence, or otherwise, solve for $0 < \theta$, π , cos $2\theta - 2 \sin 2\theta = -1.5$, rounding your answers to 2 decimal places. E/P E/P (4 marks) (4 marks) 8 Solve the following equations, in the intervals given in brackets. a 6 sin x + 8 cos x = $5\sqrt{3}$, [0, 360°] b 2 cos $3\theta - 3 \sin 3\theta = -1$, [0, 90°] c 8 cos $\theta + 15 \sin \theta = 10$, [0, 360°] x x d 5 sin $-12 \cos^2 = -6.5$, [-360° , 360°] 2 2 9 a Express 3 sin $3\theta - 4 \cos 3\theta$ in the form R sin ($3\theta - \alpha$), with R > 0 and 0 < α < 90° . (3 marks) b Hence write down the minimum value of $3 \sin 3\theta - 4 \cos 3\theta$ and the value of θ at which it occurs. (3 marks) to be found. (3 marks) to be found. (3 marks) b Hence find the value of θ at which it occurs. (3 marks) b Hence find the value maximum and minimum values of 5 sin $2\theta - 3 \cos 2\theta + 6 \sin \theta \cos \theta = -1$ for $0 < \theta$, 180° , rounding your answers to 1 decimal place. P (4 marks) c Solve 5 sin $2\theta - 3 \cos 2\theta + 6 \sin \theta \cos \theta = -1$ for $0 < \theta$, 180° , rounding your answers to 1 decimal place. P (4 marks) c Solve 5 sin $2\theta - 3 \cos 2\theta + 6 \sin \theta \cos \theta = -1$ for $0 < \theta$, 180° , rounding your answers to 1 decimal place. $< \alpha < 90^{\circ}$, and correctly solved the equation in sin θ . b Show that the correct quadratic equation is 10 sin 2 $\theta - 4 \sin \theta - 5 = 0$. c Solve this equation, for $0 < \theta$, 360°. d Explain why not all of the $\cos \theta$. (2 marks) b Hence solve the equation 12 $\cos 2\theta - 5 \sin 2\theta = -6.5$ for $0 < \alpha$, 180°. (5 marks) 7.6 Proving trigonometric identities to prove other identities to prove other identities. Example 18 θ 1 a Show that 2 sin $\cos \theta = \sin 2\theta \cdot 2 \cdot 2 \cdot 2 \cdot b$ Show that 1 + $\cos 4\theta = 2 \cos 2\theta$. a sin 2A = 2 sin A $\cos A \theta \theta \sin \theta = 2 \sin 2\theta$. $\cos 22\theta \theta$ LHS = 2 sin $\cos \cos \theta 22 = \sin \theta \cos \theta 1 = \sin 2\theta 2 = \text{RHS}$ b LHS = 1 + $\cos 4\theta = 1 + 2\cos 22\theta - 1 = 2\cos 22\theta = \text{RHS}$ 186 θ Substitute A = into the formula for sin 2A. 2 Problem-solving Always be aware that the addition formulae can be altered by making a substitution. $\theta \theta$ Use the above result for 2 sin $\cos 22\theta = 2$ Remember sin $2\theta \equiv 2 \sin \theta \cos \theta$. Use cos $2A \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A - 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A + 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A + 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A + 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A + 1$ with $\overline{A} = 2\theta$. Trigonometry and modelling Example 19 2 Prove the identity tan $2\theta \equiv 2 \cos 2A + 1$ with $\overline{A} = 2\theta$. and denominator by a common term can be helpful when trying to rearrange an expression into a required form. $2 \equiv 2 \cos 4\theta + \sin 4\theta = 2 \cos 4\theta + \sin 4\theta + \sin 4\theta = 2 \cos 4\theta + \sin 4\theta + \sin 4\theta = 2 \cos 4\theta + \sin 4\theta + \sin$ Sometimes it is easier to begin with the
RHS of the identity. $\sqrt{3}$ Exercise Problem-solving Use the addition formulae. $\pi \pi$ Write the exact values of cos and sin 667F1 Prove the following identities. cos 2Aa = $\cos A - \sin A \cos A + \sin A \sin B \cos B b$ = $2 \csc 2A \sin (B - A) \sin A \cos A 1 - \cos 2\theta c$ $\theta \sin 2\theta \sec 2\theta d = \sec 2\theta d = 1 - \tan 2\theta e 2(\sin 3\theta \cos \theta + \cos 3\theta \sin \theta) = \sin 2\theta \sin 3\theta \cos 3\theta f = - = 2 \cos \theta \sin \theta g \csc \theta - 2 \cot 2\theta \cos \theta = 2 \sin \theta \sec \theta - 1 \theta h = = 4 \cos 2x 187 \text{ Chapter 7 P E/P 2 Prove the identities: a sin (A + 60°) + sin (A - 60°) = sin A \cos A sin A + 50° = 2 sin A \cos A sin A + 50° = 2 sin B \cos B \sin (x + y) c = \cos x \cos y = \tan x + \tan y \cos (x + y) d = + 1 = \cot x \cot y \sin x \sin y = \pi \pi e \cos (\theta + -) + \sqrt{3} \sin \theta = \sin (\theta + -) 3 6 \cot A \cot B - 1 f \cot (A + B) = 0$ 3 a Show that $\tan \theta + \cot \theta \equiv 2 \csc 2\theta$. (3 marks) 4 a Show that $\sin 3\theta \equiv 3 \sin \theta \cos 2\theta - \sin 3\theta$. (3 marks) b Hence find the value of $\tan 75^\circ + \cot 75^\circ$. E/P (2 marks) b Show that $\cos 3\theta \equiv \cos 3\theta - 3 \sin 2\theta \cos \theta$. 3 $\tan \theta - \tan 3\theta$ c Hence, or otherwise, show that $\tan 3\theta \equiv 2 \cosh \theta + 3 \sin \theta + \cos \theta$. 1 – 3 tan2 θ (3 marks) (4 marks) _ 10 $\sqrt{2}$ 1 d Given that θ is acute and that cost $\theta = 3$, show that $\cos \theta = 3$, cos (x ± β) to find maximum or minimum values. Example 21 The cabin pressure, P, in pounds per square inch (psi) on an aeroplane at cruising altitude can be modelled by the equation P = 11.5 - 0.5 sin (t - 2), where t is the time in hours since the cruising altitude was first reached, and angles are measured in radians. a State the maximum and the minimum cabin pressure. b Find the time after reaching cruising altitude that the cabin first reaches a maximum pressure. c Calculate the cabin pressure after 5 hours at a cruising altitude. d Find all the times during the first 8 hours of cruising that the cabin pressure would be exactly 11.3 psi. a Maximum pressure = 11.5 - 0.5 × (-1) = 12 psi Minimum pressure = $11.5 - 0.5 \times 1 = 11$ psi b $11.5 - 0.5 \sin(t - 2) = 12 - 0.5 \sin(t - 2) = -1 - 1 < \sin(t - 2) = -1 = -1 < \sin(t - 2) = -1$ radians. t = 0.43 hours = 26 min Multiply 0.43 by 60 to get the time in minutes. c P = 11.5 - 0.5 sin (t - 2) = 11.5 - 0.070... = 11.43 psi Substitute t = 5. y Online x Explore the solution to this modelling problem graphically using technology. 189 Chapter 7 d 11.5 - 0.5 sin (t - 2) = 11.3 - 0.5 sin (t - 2) = -0.2 sin (t - 2) = 0.4 t - 2 = -3.553... 0.4115..., 2.73..., 6.6947... t = 2.41 hours, 4.73 hours. t = 2 h 25 min, 4 h 44 min Exercise P P Set the model equal to 11.3. Use sin-1(0.4) to find the principal solutions in the range 0 < t < 8.0 < t height, h, of a buoy on a boating lake can be modelled by h = 0.25 sin (1800t)°, where h is the height in metres above the buoy's resting position and t is the time in minutes. a State the maximum height the buoy is first at a height of 0.1 metres. c Calculate the time interval between successive minimum heights of the buoy. 2 The angle of displacement of a pendulum, θ , at time t seconds after it is released is modelled as $\theta = 0.03 \cos(25t)$, where all angles are measured in radians. a State the maximum displacement of the pendulum according to this model. b Calculate the angle of displacement of the pendulum has a displacement of 0.01 radians. P 3 The price, P, of stock in pounds during a 9-hour trading window can be modelled by P = 17.4 + 2 sin (0.7t - 3), where t is the time in hours after the stock market opens, and angles are measured in radians. a State the beginning and end price of the stock when it firsts shows a profit of £0.40 above the day's starting price. At what time should the trader sell the stock? P 4 The temperature of an oven can be modelled by the equation T = 225 - 0.3 sin (2x - 3), where T is the temperature of the oven. b Find the times during the first 10 minutes when the oven is at a minimum temperature. c Calculate the time when the oven first reaches a temperature of 225.2 °C. E/P 5 a Express 0.3 sin $\theta - 0.4 \cos \theta$ in the form R sin ($\theta - \alpha$)°, where R > 0 and 0 < α < 90°. Give the value of α to 2 decimal places. (4 marks) 190 Trigonometry and modelling b i Find the maximum value of α to 2 decimal places. $0.3 \sin \theta - 0.4 \cos \theta$. (2 marks) ii Find the value of θ , for $0 < \theta < 180$ at which the maximum occurs. (1 mark) Jack models the temperature in his house, T °C, on a particular day by the equation T = 23 + 0.3 sin (18x)° - 0.4 cos (18x)°, x > 0 where x is the number of minutes since the thermostat was adjusted. c Calculate the minimum value of T predicted by this model, and the value of x, to 2 decimal places, when this minimum occurs. (3 marks) d Calculate, to the nearest minute, the times in the first hour when the temperature is predicted, by this model, to be exactly 23 °C. (4 marks) $E/P \pi 6$ a Express 65 cos $\theta - 20 \sin \theta$ in the form R cos ($\theta + \alpha$), where R > 0 and 0 < $\alpha < 2$ Give the value of α correct to 4 decimal places. (4 marks) π 7 a Express 200 sin θ – 150 cos θ in the form R sin (θ – α), where R > 0 and 0 < α < 2 Give the value of α to 4 decimal places. (4 marks) A city wants to build a large circular wheel as a tourist attraction. The height of a tourist on the circular wheel is modelled by the equation H = 70 – 65 cos 0.2t + 20 sin 0.2t where H is the height of the tourist above the ground in metres, t is the number of minutes after boarding and the angles are given in radians. Find: b the maximum height of the wheel (2 marks) c the time for one complete revolution. (4 marks) c the time for one complete revolution. (4 marks) c the time for one complete revolution. marks) E/P The electric field strength, E V/m, in a microwave of width 25 cm can be modelled using the equation $4\pi x 4\pi x E = 1700 + 200 \sin() - 150 \cos() + 200 \sin() - 150 \cos() + 200 \sin() + 2$ where this maximum occurs. (3 marks) c Food in the microwave will heat best when the electric field strength at the centre of the food is above. For food of the same type and mass, the energy transferred by the oven is proportional to the square of the electric field strength. Given that a square of chocolate placed at a point of maximum field strength takes 20 seconds to melt. b State two limitations of the model. 191 Chapter 7 Mixed exercise P 7 1 a Without using a calculator, find the value of: $1_1 cos 15^\circ - cos 40^\circ sin 10^\circ ii \sqrt{2} \sqrt{2} 1 - tan 15^\circ iii \dots 1 + tan 15^\circ P P \sqrt{5} + 11_where x is acute and that cos (x - y) = sin y, show that tan y = 2 \sqrt{5} 3$ The lines 11 and 12, with equations y = 2x and 3y = x - 1 respectively, are drawn on the same set of axes. Given that the angles 11 and 12 make with the positive x-axis are A and B respectively, a write down the same on both axes and that the angles 11 and 12 make with the angle between 11 and 12. P P P 4 In $\triangle ABC$, AB = 5 cm and $_AC = 4$ cm, $\angle ABC = (\theta - 30^\circ)$ and $\angle ACB = (\theta + 30^\circ)$. Using the sine rule, show that tan $\theta = 3\sqrt{3}$. $_5$ The first three terms of an arithmetic series are $\sqrt{3} \cos \theta$, sin $(\theta - 30^\circ)$ and sin θ , where θ is acute. Find the value of θ . 6 Two of the angles, A and B, in $\triangle ABC$ are such that tan $A = _4$, tan B = -4. 12 3 a Find the exact value of: i sin (A + B) 5 ii tan 2B. b By writing C as 180° – (A + B), show that cos C = -65P3332 and cos y = 7The angles such that sin x = $\sqrt{5}\sqrt{10}3$ a Show that cos 2x = -5 b Find the value of cos 2y. c Show without using your calculator, that: π i tan (x + y) = 7 ii x - y = 4P8Given that sin x cos y = 2 and cos x sin y = 3, 1 1 a show that sin (x + y) = 5 sin (x - y). Given also that tan $\overline{y} = k$, express in terms of k: b tan x c tan $2x E/P _ 1_ 9$ a Given that $\sqrt{3} \sin 2\theta + 2 \sin 2\theta = 1$, show that tan $2\theta = _\sqrt{3}_$ b Hence solve, for $0 < \theta < \pi$, the equation $\sqrt{3} \sin 2\theta + 2 \sin 2\theta = 1$. 192 (2 marks) (4 marks) Trigonometry and modelling E/P E/P 10 a Show that cos $2\theta = 5 \sin \theta$ may be written in the form a sin $2\theta + b \sin \theta + c = 0$, where a, b and c are constants to be found. (3 marks) b Hence solve, for $-\pi < \theta < \pi$, the equation cos $2\theta = 5 \sin \theta$. (4 marks) 1 _ 11 a Given that $2 \sin x = \cos (x - 60)^\circ$, show that tan $x = ___4 - \sqrt{3}$ (4 marks) b Hence solve, for $0 < x < 360^\circ$, 2 $\sin x = \cos (x - 60^\circ)$, giving your answers to 1 decimal place. E/P 12 a Given that 4 sin (x + 70^\circ) = cos (x + 20^\circ), show that $\tan x = -5 \tan 70^\circ$. 3 b Hence solve, for $0 < x < 180^\circ$, $4 \sin (x + 70^\circ) = cos (x + 20^\circ)$, giving your answers to 1 decimal place. P (2 marks) (3 marks) 13 a Given that α is acute and $\tan \alpha = -4$, prove that 3 3 sin ($\theta + \alpha$) + 4 cos ($\theta + \alpha$) = 5 cos θ b Given that sin x = 0.6 and cos x = -0.8, evaluate cos (x + 270°) and cos (x + 540°). E/P 14 a Prove, by counter-example, that the statement sec (A + B) = sec A + sec B, for all A and B is false. nn b Prove that tan θ + cot θ = 2 cosec 2 θ , $\theta \pm$, n \in . 2 P E/P (2 marks) (4 marks) 2 tan θ with an appropriate value of θ , 15 $1 - \tan 2\theta = \pi a$ show that $\tan = \sqrt{2} - 1.83\pi b$ Use
the result in a to find the exact value of $\tan = 8 = 16a$ Express sin $x - \sqrt{3} \cos x$ in the form R sin $(x - \alpha)$, with R > 0 and $0 < \alpha < 90^\circ$. (4 marks) b Hence sketch the graph of $y = \sin x - \sqrt{3} \cos x$, for $-360^\circ < x < 360^\circ$, giving the coordinates of all points of intersection with the axes. (4 marks) E/P π 17 Given that 7 cos 2 θ + 24 sin 2 θ = R cos (2 θ - α), where R > 0 and 0 < α < ____, find: 2 a the value of R and the value of α , to 2 decimal places b the maximum value of 14 cos 2 θ + 48 sin θ cos θ . (4 marks) (1 mark) c Solve the equation 7 cos 2 θ + 24 sin 2 θ = 12.5, for 0 < θ < 180°, giving your answers to 1 decimal place. (5 marks) 193 Chapter 7 E/P π 18 a Express 1.5 sin 2x + 2 cos 2x in the form R sin (2x + α), where R > 0 and 0 < α < , 2 giving your values of R and α to 3 decimal places where appropriate. (4 marks) b Express 3 sin x cos x + 4 cos 2 x in the form a sin 2x + b cos 2x + c, where a, b and c are constants to be found. (3 marks) c Hence, using your answer to part a, deduce the maximum value of 3 sin x cos x + 4 cos 2 x. E/P E/P (1 mark) θ 19 a Given that sin 2 = 2 sin θ , show that $\sqrt{17}$ sin ($\theta + \alpha$) = 1 and state the value of α . 2 θ b Hence, or otherwise, solve sin 2 = 2 sin θ for 0 < θ < 360°. 2 (4 marks) 21 Using known trigonometric identities, prove the following: $\pi \pi$ a sec θ $\cos \theta = 2 \csc 2\theta$ b tan $(-x) = 2 \tan 2x 4 4 c \sin (x + y) \sin (x - y) = \cos 2 y - \cos 2 x E/P$ (4 marks) 20 a Given that 2 cos $\theta = 1 + 3 \sin \theta$, show that R cos $(\theta + \alpha) = 1$, where R and α are constants to be found. (2 marks) b Hence, or otherwise, solve 2 cos $\theta = 1 + 3 \sin \theta$ for $0 < \theta < 360^\circ$. P (3 marks) d $1 + 2 \cos 2\theta + \cos 4\theta = 4 \cos 2\theta$ $\theta \cos 2\theta 1 - \cos 2x 22$ a Use the double-angle formulae to prove that _____ = tan 2 x. 1 + cos 2x (4 marks) 1 - cos 2x b Hence find, for $-\pi < x < \pi$, all the solutions of _____ = 3, leaving your answers 1 + cos 2x in terms of π . (2 marks) E/P 23 a Prove that cos 2 θ - sin 2 θ = cos 4 θ . (4 marks) 1 b Hence find, for 0 < θ < 180°, all the solutions of $\equiv \tan \theta$. sin 2 θ (4 marks) b Verify that $\theta = 180^\circ$ is a solution of the equation sin 2 $\theta = 2 - 2 \cos 2\theta$. (1 mark) c Using the result in part a, or otherwise, find the two other solutions, $0 < \theta < 360^\circ$, of the equation sin 2 $\theta = 2 - 2 \cos 2\theta$. (3 marks) E/P 25 The curve on an $\cos 4 2\theta - \sin 4 2\theta = 2 \text{ E/P} (2 \text{ marks}) 1 - \cos 2\theta 24 \text{ a Prove that}$ oscilloscope screen satisfies the equation $y = 2 \cos x - \sqrt{5} \sin x$. a Express the equation of the curve in the form $y = R \cos (x + \alpha)$, where R and $\pi \alpha$ are constants and R. 0 and $0 < \alpha$, _ 2 b Find the values of x, 0 < x, 2π , for which y = -1. 194 (4 marks) (3 marks) Trigonometry and modelling E/P 26 a Express 1.4 sin $\theta - 5.6 \cos \theta$ in the form R sin (θ - α), where R and α are constants, R > 0 and 0 < α < 90°. Round R and α to 3 decimal places. (4 marks) b Hence find the maximum occurs. (3 marks) The length of daylight, d(t) at a location in northern Scotland can be modelled using the equation 360t ° 360t ' d(t) = 12 + 5.6 cos (___) + 1.4 sin (___) 365 365 where t is the numbers of days into the year. c Calculate the minimum number of daylight hours in northern Scotland as given by this model. (2 marks) d Find the value of t when this minimum number of daylight hours in x + 5 cos x in the form R sin (x + α), where R and α are constants, R > 0 and $0 < \alpha < 90^{\circ}$. Round α to 1 decimal place. A runner's speed, v in m/s, in an endurance race can be modelled by the equation 50 v (x) = $0 < x < 300 2x^{\circ} 2x^{\circ} 12 \sin() + 5 \cos() 55$ where x is the time in minutes since the beginning of the race. b Find the minimum value of v. c Find the time into the race when this speed occurs. (1 mark) (4 marks) (2 marks) (1 mark) Challenge 1 Prove the identities: $\cos 2\theta + \cos 4\theta$ a $\equiv -\cot \theta \sin 2\theta - \sin 4\theta$ b $\cos x + 2 \cos 3x + \cos 5x \equiv 4 \cos 2x \cos 3x + 2 \cos 3x + \cos 5x \equiv 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 3x + \cos 5x = 4 \cos 2x \cos 5x = 4 \cos 5x =$ 1 O D Hint Find expressions for $\angle BOD$ and AB, then consider the lengths OD and DB. C Use this construction to prove: $a \sin 2\theta \equiv 2 \cos 2\theta - 1$ 195 Chapter 7 Summary of key points 1 The addition (or compound-angle) formulae are: • sin (A + B) \equiv sin A cos B + cos A sin B sin (A - B) \equiv sin A cos B - cos A sin B tan A + tan B $1 - \tan A \tan B \tan A - \tan B \tan (A - B) \equiv$ $1 + \tan A \tan B \cdot \cos (A + B) \equiv \cos A \cos B - \sin A \sin B \cos (A - B) \equiv \cos A \cos B + \sin A \sin B \cos (A - B) \equiv \cos A \cos A = \sin A \sin B \cos (A - B) \equiv \cos A \cos B + \sin A \sin B \cos (A - B) \equiv \cos A \cos B + \sin A \sin B \cos (A - B) = \cos A \cos (A - B) \equiv \cos A \cos (A - B) =
\cos$ can write any expression of the form a sin θ + b cos θ as either: • R sin ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm b$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$), with R > 0 and 0 < α < 90° or • R cos ($\theta \pm a$). Cartesian form by substitution \rightarrow pages 198-202 \bigcirc Convert parametric equations into Cartesian form using \rightarrow pages 202-206 trigonometric equations of curves \bigcirc Solve coordinate geometry problems involving parametric \rightarrow pages 209-213 equations \bigcirc Use parametric equations in modelling in a variety of contexts \rightarrow pages 213-220 Prior knowledge check 1 Rearrange to make t the subject: c y = 2 - 4 ln t a x = 4t - kt b y = 3t2 d x = 1 + 2e-3t \leftarrow GCSE Mathematics; Year 1, Chapter 14 2 3 Parametric equations can be used to describe the path of a ski jumper from the point of leaving the ski ramp to the point of landing. \rightarrow Exercise 8E, Q8 Write in terms of powers of cos x: b sin 2x a 4 + 3 sin 2x c cot x d 2 cos x + cos 2x \leftarrow Section 7.2 State the ranges of the following functions. a y = ln (x + 1), x > 0 b y = 2 sin x, 0 < x < \pi 1 d y = ______, x > -2 c y = x2 + 4x - 2, -4 < x < 1 2x + 5 \leftarrow Year 1, Chapter 6 4 A circle has centre (0, 4) and radius 5. Find the coordinates of the points of intersection of the circle and the line with equation - Year 1, Chapter 6 2y - x - 10 = 0. 197 Chapter 8 8.1 Parametric equations You can write the x- and y-coordinates of each point on a curve as functions of a third variable. This variable is called a parameter and is often represented by the letter t. defined using parametric equations x = p(t) and y = q(t). Each value of the parameter, t, defines a point on the curve with coordinates (p(t), q(t)). y 6 2 x = t + 1, y = 2t, t > 0 t 5 4 3 2 1 - 1 0 - 1 1 2 3 4 5 6 x Watch out The value of the parameter t is generally not equal to either the x- or the y-coordinate, and more than one point on the curve with coordinates (p(t), q(t)). y 6 2 x = t + 1, y = 2t, t > 0 t 5 4 3 2 1 - 1 0 - 1 1 2 3 4 5 6 x Watch out The value of the parameter t is generally not equal to either the x- or the y-coordinate, and more than one point on the curve with coordinates (p(t), q(t)). the same x-coordinate. You can convert between parametric equations of the parameter tells you the values of t you would need to substitute to find the coordinates of the points on the curve. 22 + 1 When t = 2, x = 2.5 and $y = 2 \times 2 = 4.2$ This corresponds to the point (2.5, 4). 0.52 + 1 When t = 0.5, x = 2.5 and y = 2 \times 0.5 = 1.0.5 This corresponds to the point (2.5, 1). Notation A Cartesian equation in two dimensions involves the variables x and y only. You can use the domain and range of the parametric functions to find the domain and range of the resulting Cartesian function. \blacksquare For parametric equations x = p(t) and y = q(t) with Cartesian equation of f(x) is the range of $p(t) \oplus$ the range of p(t) \oplus the range of $p(t) \oplus$ the range of p(t) \oplus the range of $p(t) \oplus$ the range of p(t) \oplus the ran domain and range of f(x). c Sketch the curve within the given domain for t. x a x = 2t so t = 2y = t2 (1) (2) Substitute (1) into (2): x 2 x 2 y = ($_) = 24198$ A Cartesian equation only involves the variables x and y, so you need to eliminate t. Rearrange one equation into the form t = ... then substitute into the other equation. This is a quadratic curve. Parametric equations b x = 2t, -3, t, 3 So the domain of f(x) is -6, x, 6. y = t2, -3, t, 3 So the range of f(x) is 0 < y, 9. c y = x = 2t over the domain of f is the range of x = 2t over the domain -3 < t < 3 is -6 < x < 6. \leftarrow Section 2.1 The range of f is the range of the parametric function for x. The range of x = 2t over the domain -3 < t < 3 is -6 < x < 6. \leftarrow Section 2.1 The range of f is the range of the parametric function for x. Choose your inequalities carefully. y = t2 can equal 0 in the interval -3 < t < 3, so use k where k is a constant to be found. $x = \ln(t + 3)$, b Write down the range of f(x). a $x = \ln(t + 3)$ ex = t + 3 So ex -3 = t Substitute t = ex -3 into 1 1 y = _____ = ____ t + 5 ex - 3 + 5 1 = ex + 2 When t = -2: x = ln (t + 3) = ln 1 = 0 As t increases ln (t + 3) increases, so the range of the parametric function for x is x > 0. The Cartesian equation is 1 y = ______, x>0 ex + 2 1 1 b When t = -2: y = ______ = _____ t+5 3 As t increases, so the range 1 of the parametric function for y is y < ______ 3 1 The range of f(x) is y < ______ 3 y Online x Sketch this parametric curve using technology. ex is the inverse function of $\ln x$. Rearrange the equation for x into the form t = ... then substitute into the equation for y. To find the domain for f(x), consider what value x takes when t = -2 and what happens when t increases. The range of y alues y can take within the given range of the parameter. You could also find the range of f(x) by considering the domain of f(x). In each case find the range of f(x) is $y < 13 \leftarrow$ Section 2.1 199 Chapter 8 Exercise 8A 1 Find a Cartesian equation for each of these parametric equations, giving your answer in the form y = f(x). In each case find the domain and range of f(x). a x = t - 2, y = t2 + 1, 1 c x = t + 2, y = t2 + 1, 1 c x = t + 1, 1 f x = ____, t+1 t \in \mathbb{R} If the domain of t is given as t $\neq 0$, this implies that t can take any value in other than 0. 1 e x = ____, t-2 t>0 1 y = _____, t-2 t>0 1 y = ____, t-2 t>0 1 y = find a Cartesian equation for the curve in the form y = f(x) giving the domain on which the curve is defined ii find the range of f(x). 1 b x = ln (t + 3), y = (t - 5), t < 4 t + 5 c x = et, P y = e3t, t $\in \mathbb{R}^-$ 3 A curve C is defined by the parametric equations $x = \sqrt{t}$, y = t(9 - t), 0 < t < 5. a Find a Cartesian equation of the curve in the form y = f(x), and determine the domain and range of f(x). b Sketch C showing clearly the Problem-solving coordinates of any turning y = t(9 - t) is a quadratic with a negative t2 term and roots at points, endpoints and intersections t = 0 and t = 9. It will take its maximum value when t = 4.5. with the coordinate axes. 4 For each of the following parametric curves: i find a Cartesian equation for the curve in the form y = f(x) ii find the domain and range of f(x) iii sketch the curve within the given domain of t. a $x = 2t^2 - 3$, c x = t + 1, $y = 9 - t^2$, 1 y = 1, t - 2, b x = 3t - 1, y = (t - 1)(t + 2), -4 < t < 4 t>0 to Parametric equations P 5 The curves C1 and C2 are defined by the following parametric equations. t 1 C2: x = 1 + 2t, y = 2 + 3t 2 < t < 5 2t - 3 2t - 3 a Show that both curves are segments of the same straight line. Notation Straight lines and line segments. E/P 2 k where k is a constant to be found. b Write down the range of f(x). E/P 8 A diagram shows a curve C with parametric equations of the 2π 8 m others to check: 8 cos (20 ×) = 8 cos () = -4153 Parametric equations c The figure skater crosses the y-axis when $x = 0, 0 = 8 \cos t 0 = \cos t \pi 3\pi 5\pi 7\pi$ So, _, _ , _ , ... 2 2 2 Substitute these t-values into y. $\pi t = _ : 2 \pi \pi \pi 1 y = 12 \sin (_ x _ - _) = 12 = 12 = 12 = 12 = 12 = 12 =$ -) = 12 sin () 2 2 12 3 = 3.11 (2 d.p.) $7\pi t$ = : 2 17 π 1 7 $\pi \pi y$ = 12 sin (× -) = 12 sin (× -) = 12 sin () 2 2 12 3 = 11.59 (2 d.p.) So the skater crosses the y-axis at (0, -3.11), (0, -11.59). 2 π d The period of x = 8 cos 20t is , 20 so the skater returns to his x-position 4π 2 π after min, min, ... 20 20 2 π π The period of y = 12 sin (10t - _) is ___, 3 10 so the skater returns to his y-position $4\pi 2\pi$ after ___ min, ___ min, ... 10 10 So the skater first completes a full figure-of-eight motion after 2π ____ mins = 0.628 ... mins or 38 seconds 10 (2 s.f.). Find solutions to 8 cos t = 0 in the domain t > 0. There are 4 points of intersection so consider the first 4 solutions, and check that these give different values of y. Use your calculator to find the corresponding values of y. You can give your answers as decimals ___ I or as exact values: 12 sin (- __) = -3√6 + 3√6 12 y Online x Find points of intersection of this curve with the coordinate axes using technology. Check that these look sensible from the graph. The motion of the skater appears to be symmetrical about the x-axis so these look right. 2π The period of a cos (bx + c) is and the periods. This occurs at the least common multiple of the two periods. 217 and the period b 2π of a sin (bx + c) is b Problem-solving In order for
the figure skater to return to his starting position, both parametric equations must complete full periods. Chapter 8 Exercise P 8E 1 A river flows from north to south. The position at time t seconds of a rowing boat crossing the river from west to east is modelled by the parametric equations x = 0.9 tm, y = -3.2 tm where x is the distance travelled north. Given that the river is 75 m wide, a find the time taken to get to the other side b find the distance the boat has been moved off-course due to the current c show that the motion of the boat is a straight line d determine the speed of the boat. P 2 The position of a small plane coming into land at time t minutes after it has started its descent is modelled by the parametric equations x = 80t, y = -9.1t + 3000, 0 < t < 1000329 where x is the horizontal distance travelled (in metres) and y is the vertical distance travelled (in metres) from the point of starting its descent. a Find the horizontal distance travelled (in metres) and y is the vertical distance travelled (in metres) and y is the vertical distance travelled (in metres) from the point of starting its descent. a Find the horizontal distance travelled (in metres) and y is the vertical distance travelled (in metres) from the point of starting its descent. a Find the horizontal distance travelled (in metres) and y is the vertical distance travelled (in metres) from the point of starting its descent. a Find the horizontal distance travelled (in metres) and y is the vertical distance traveled (in metres) and y is the vertical di kicked from the ground with an initial speed of 20 m s-1 at an angle of 30°. Its position after t seconds can be described using the following parametric equations $x = 10\sqrt{3}$ t m, y = (-4.9t2 + 10t) m, 0 < t < k a Find the horizontal distance travelled by the ball when it hits the ground. A player wants to head the ball when it is descending between 1.5 m and 2.5 m off the ground. b Find the range of time after the ball has been kicked at which the player can head the ball. P 4 The path of a dolphin leaping out of the water can be modelled with the following parametric equations x = 2t m, y = -4.9t2 + 10t m where x is the horizontal distance from the dolphin has started its jump. a Find the time the dolphin takes to complete a single jump. b Find the horizontal distance the dolphin travels during a single jump. c Show that the dolphin's path is modelled by a quadratic curve. d Find the maximum height of the dolphin. 218 Parametric equations $x = 12 \sin t$, $y = 12 - 12 \cos t$ where x is the horizontal distance in metres of the car from the start of the ride and y is the height in metres above ground level of the car. a Show that the motion of the car is a circle with radius 12 m. b Hence, find the maximum height of the car is a circle with radius 12 m. b Hence, find the maximum height of the car. following parametric equations $y \pi \pi x = t - 4 \sin t$, $y = 1 - 2 \cos t$, - - < t < - 2 2 P a Find the length of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (4 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (3 marks) b Given that the cross-section of the bowl. (4 marks) b Given that the crossty = 2t, x O y t>0 The diagram shows the path of the particle. a Find the distance from the origin to the particle at time t = 0.5. b Find the coordinates of the points where the particle crosses the y-axis. x O Another particle travels in the same plane with its path given by fixed origin O is given by the parametric equations $t^2 - 3t + 2$, x =the equation y = 2x + 10. c Show that the paths of these two particles never intersect. E/P 8 The path of a ski jumper from the point of leaving the ramp to the p answer in exact form. 4 A curve has parametric equations 1 = 1, 2t + 1 + 1, $2 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 + 1 = 2 \ln (t + _)$, $2t + 1 = 2 \ln (t + _)$ (1 + t)(1 - t) t>1 x2 Express t in terms of x, and hence show that a Cartesian equation of the curve is y = 2x - 17 A circle has parametric equations $x = 4 \sin t - 3$, $y = 4 \cos t + 5$, $0 < t < 2\pi$ a Find a Cartesian equation of the and (b, 0). b Find the values of a and b. P 1 6 A curve has parametric equations x = 1, 1+t 1 y = _____ circle. b Draw a sketch of the circle. c Find the exact coordinates of the points of intersection of the circle with the y-axis. E/P 8 The curve C has parametric equations 2 - 3t x = _____, 1+t E 3 + 2t y = _____ , 1+t 0 0 for every value of x in that interval. Links d2y To find the second derivative, f0(x) or 2, you differentiate twice with respect to x. $dx \leftarrow Year 1$, Chapter 12 y y O y y = ex y = 5x - x2 y = x3 - 6x2 - 9x x O O dy ___ = -2 so the curve is 2 2 dx concave for all $x \in .2 x x dy ___ = ex$ which is always 2 dy ___ =
6x - 12 so the curve is 2 2 2 dx positive, so the curve is 2 2 2 dx positive, so the curve is 2 2 2 dx positive, so the curve is 2 2 dx positive. = x3 + 4x + 3 is concave. f(x) = x3 + 4x + 3 f9(x) = 3x2 + 4 f0(x) = 6x For f(x) to be concave to convex for all $x \in .$ e2x 4e2x Problem-solving Write down the condition for a convex for all $x \in .$ e2x 4e2x Problem-solving Write down the condition for a convex function and a conclusion. The point at which a curve changes from being concave to convex (or vice versa) is called a point of inflection. The diagram shows the curve with equation $y = x^3 - 2x^2 - 4x + 5$. y In the interval [-2, 0] the curve is convex. -5 At some point between 0 and 1 the curve is convex. -5 At some point of inflection. inflection is a point at which f 0(x) changes sign. To find a point of inflection you need to show that f 0(x) = 0 at that point, and that it has different signs on either side of that point. Example 28 The curve C has equation $y = x^3 - 2x^2 - 4x + 5$. a Show that C is concave on the interval [-2, 0] and convex on the interval [1, 3]. b Find the coordinates of the point of inflection. a dy dx = dy $3x^2 - 4x - 4$ Differentiate $y = x^3 - 2x^2 - 4x + 5$ with respect to x twice. $2 = 6x - 4 dx^2 d^2y = 6x - 4 dx^2 d^2y$ -2, d 2v d 2v d 2v = -16 and when x = 0, = -4, so 0 for all 1 < x < 3. Therefore, $y = x^3 - 2x^2 - 4x + 5$ is convex on the interval [1, 3]. d 2y b 2 = 6x - 4 = 0 dx Find the point where f0(x) = 0. You have already determined that f0(x) changes sign on either side of this point. 6x = 44 2x = 6 = 3 Substitute x into y gives y = (3)-2(3) - 4(3) + 5 = 23 22 247 Online x Explore the solution to this example graphically using technology. 27 So the point of inflection of the curve C 47 2 is (_3, 27). Exercise P 9I 1 For each of the following functions, find the interval on which the function is: i convex P ii concave a f(x) = x3 - 3x2 + x - 2 b f(x) = x4 - 3x3 + 2x -1 c f(x) = sin x. $0 < x < 2\pi d f(x) = e f(x) = f(x) = ln x$. $x > 0 - x^2 + 3x - 7 ex - x^2 2 f(x) = arcsin x$. -1 < x < 1 a Show that f(x) is convex on the interval (0, 1). c Hence deduce the point of inflection of f. P.3 Find any point(s) of inflection of the following functions, $x_3 - 2x_2 + x - 1 a f(x) = cos^2 x$. $-2 \sin x$, $0 < x < 2\pi b f(x) =$ $x \neq 2 x - 2 3 x$, $x \neq 2 d f(x) = \arctan x c f(x) = x^2 - 4 P 4 f(x) = 2x^2 \ln x$, x > 0 Show that f has exactly one point of inflection and determine its nature. b Find the coordinates of any non-stationary points of inflection on C. 259 Chapter 9 P 6 The curve C has equation y = xex. a Find the exact coordinates of the stationary point on C and determine its nature. Problem-solving b Find the coordinates of any non-stationary points of inflection on C. 259 Chapter 9 P 6 The curve C has equation y = xex. a Find the exact coordinates of the stationary point on C and determine its nature. negative values of x. c Hence sketch the graph of y = xex. P 7 For each point on the graph, state whether: i f9(x) is positive, negative or zero ii f0(x) is positive, negative or zero ii f0(x) is positive, negative or zero ii f0(x) is positive, negative or zero y y = f(x) A C D O x B P E E/P $\pi \pi 8 f(x) = \tan x$, - - < x < 2 2 Prove that f(x) has exactly one point of inflection, at the origin. 9 Given that y = x(3x - 1)5, dy d 2y a find _____ 2 dx dx (4 marks) b find the points of inflection of y. (4 marks) 10 A student is attempting to find the points of inflection on the curve C with equation <math>y = 4(x - 5) 2 When $dx^2 2y d = 12(x - 5) 2(x - 5)$ of inflection at x = 5. 260 a Identify the mistake made by the student. (2 marks) b Write down the coordinates of the stationary point on C and determine its nature. (2 marks) Differentiation E/P 11 A curve C has equation $y = 3 \times 2 \ln x - 2x + 5$, $x > 0 \times 1_3$ (5 marks) Show that the curve C is convex for all x > e - 2. Challenge 1 Prove that every cubic curve has exactly one point of inflection. 2 The curve C has equation y = ax4 + bx 3 + cx2 + dx + e, $a \neq 0$ a Show that C has at most two points of inflection. 9.10 Rates of change \blacksquare You can use the chain rule to connect rates of change in situations involving more than two variables. Example 29 Given that the area of a circle A cm2 is related to its radius r cm by the formula A = πr^2 , and that dr dA the rate of change of its radius in cm s-1 is given by __ = 5, find __ when r = 3. dt dt A = πr^2 . $dA = dr dA Using __ = dt dA __ = dt Problem-solving 2\pi r dA __ dr dr × __ dt 2\pi r × 5 = 30\pi$, when r = 3. In order to be able to apply the chain rule to dA dA find ______ you need to know ____. You can find it by dt dr differentiating A = πr^2 with respect to r. You should use the chain rule, giving the derivatives. Example 30 The volume of a hemisphere V cm3 is related to its radius r cm by the formula V = $3 \pi r^3$ and the total surface area S cm2 is given by the formula S = $\pi r^2 + 2\pi r^2 = 3\pi r^2$. Given that the rate of dV dS increase of volume, in cm3 s-1, ____ = 6, find the rate of increase of surface area _____ dt dt 2 2 3 2 V = ____ 3 \pi r and S = $3\pi r dV$ _____ dr dS = $2\pi r^2$ and _____ = $6\pi r dr$ This is area of circular base plus area of curved surface. dV dS As V and S are functions of r, find _____dr dr 261 Chapter 9 dS dS dr dV Now ___ = ___ × ___ × ____dt dr dV dt 1 = 6 m × ____ 2 × 6 2 m Use the chain rule together with the property dV dr that ____ = 1 ÷ ____ dV dr 18 = ____ r An equation which involves a rate of change is called a differential equation. and Links You can use integration to solve differential equations. -> Section 11.10 Example 31 In the decay of radioactive particles and let t be time. The rate of change of the number of dN particles is proportional to N. dt dN i.e. = -kN, where k is a positive constant. dt The minus sign arises because the number of particles is decreasing. dN dN \propto N so you can write = kN dt dt where k is the constant of proportion. Example 32 Newton's law of cooling states that the rate of loss of

temperature of a body is proportional to the excess temperature of the body over its surroundings. Write an equation that expresses this law. Let the temperature of the temperature of the surroundings. $d\theta$ i.e. $-k(\theta - \theta 0)$, where k is a positive dt constant. 262 $\theta - \theta 0$ is the difference between the temperature of the body and that of its surroundings. The minus sign arises because the temperature is decreasing. The minus sign arises because the temperature of the body and that of its surroundings. proportional to its 4 surface area. Assuming that the head is spherical, that the volume of a sphere is _3 πR3 cm3 and that the surface is 4πR2 cm2, write down a differential equation for the rate of change of radius of the snowman's head. dV The first sentence tells you that _____ = -kA, dt where V cm3 is the volume, t seconds is time, k is a positive constant and A cm2 is the surface area of the snowman's head. Since $4 \ 3 \ V = 3 \ \pi R \ dV$ dR dt dt The question asks for a differential equation in terms of R, so you need to use the expression for V in terms of R. The chain rule is used here because this is a related rate of change. dV But as = -kA dt Use the expression for A in terms of R. dR 4 π R2 × $= -k \times 4\pi$ R2 dt Divide both sides by the common factor 4π R2. \therefore Exercise dR dt = -k This gives the rate of change of radius as required. 9J P dr dA 1 1 Given that A = 4 \pi2 and that = 6, find when r = 2. dt dt P dy dx 2 Given that y = xex and that = 5, find when x = $\overline{2}$. dt dt P dr d $\theta \pi$ 3 Given that $r = 1 + 3 \cos \theta$ and that r = 3, find when $\theta = -6$ dt dt P dV dr 1 4 Given that $V = -3 \pi 3$ and that r = 3, find when r = 3, find when r = 3, find when $\theta = -6$ dt dt P dV dr 1 4 Given that $V = -3 \pi 3$ and that r = -3, find when $\theta = -6$ dt dt P dV dr 1 4 Given that $V = -3 \pi 3$ and that r = -3, find when $\theta = -6$ dt dt P dV dr 1 4 Given that $V = -3 \pi 3$ and that r = -3, find when $\theta = -6$ dt dt P dV dr 1 4 Given that $V = -3 \pi 3$ and that r = -3, find when r =y = f(x), y > 0. At any point P on the curve, the gradient of C at A is 2 dy xy Show that = dx 16 263 Chapter 9 P 7 Liquid is pouring into a constant rate of 30 cm 3 s-1. At time t seconds liquid is 2 3 -1 3 leaking from the container at a rate of ______15 V cm s, where V cm is the volume of the liquid in the container at that time. dV Show that -15 _____ = 2V - 450. dt P 8 An electrically-charged body loses its charge, Q coulombs, at a rate, measured in coulombs per second, proportional to the charge Q. Write down a differential equation in terms of Q and t where t is the time in seconds since the body started to lose its charge. P 9 The ice on a pond has a thickness x mm at a time t hours after the start of freezing. The rate of increase of x is inversely proportional to the square of x. Write down a differential equation in terms of x and t. P 10 The rate of increase of x is inversely proportional to the square of 0.4 cm per second. dC a Find , where C is the circle, and interpret this value in the context of dt the model. b Find the ratius of the circle when its area is increasing at the rate of 20 cm2 per second. P 11 The volume of a cube is decreasing at a constant rate of 4.5 cm3 per second. Find: a the rate at which the length of one side of the cube is decreasing at the rate of 2 mm per second. P 12 Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, t > 0, the volume of fluid remaining in the tank is V m3. The rate at which the fluid flows in m3 min-1 is proportional to the square root of V. dh Show that the depth, h metres, of fluid in the tank satisfies the differential equation $= -k\sqrt{h}$, dt where k is a positive constant. P 13 At time, t seconds, the surface area of a cube is A cm2 and the volume is V cm3. The surface area of the cube is expanding at a constant rate of 2 cm2 s-1. a Write an expression for V in terms of A. dV b Find an expression for V in terms of A. dV b Find an expression for U in terms of A. dV b Find an expression for L a Write and the angle of the cube is expanding at a constant rate of 2 cm2 s-1. a Write an expression for V in terms of A. dV b Find an expression for V in cone between the slanting edge and the vertical is 30°, show that 1 the volume of the salt is _9 πh3, where h is the height of salt at time t seconds. Show that the rate of change of the height of the salt in the funnel is inversely proportional to h2. Write down a differential equation relating h and t. 264 Differentiation Mixed Exercise E E/P E 9 1 Differentiate with respect to x: a ln x2 (3 marks) b (4 marks) x2 sin 3x dy 2 a Given that $2y = x - \sin x \cos x$, $0 < x < 2\pi$, show that $y = \sin 2 x$. dx b Find the coordinates of the points of inflection of the curve. 3 Differentiate, with respect to x: sin x a x, x > 0 (4 marks) x, $x \in 4$ f(x) = $2 \times +2$ a Given that f(x) is increasing on the interval [-k]k], find the largest possible value of k. (4 marks) b Find the exact coordinates of the points of inflection of f(x). E/P (4 marks) (5 marks) 5 The function of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E/P E$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P E/P$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P$ a Find the set of values of x by $3 f(x) = 12 \ln x + x 2 E/P E/P$ a Find the set of values of x by b Find the coordinates of the point of inflection of the function f. (4 marks) 6 Given that a curve has equation $y = \cos 2 x + \sin x$, $0 < x < 2\pi$, find the coordinates of the stationary points of the curve. (6 marks) _____7 The maximum point on the curve with equation $y = x\sqrt{\sin x}$, $0 < x < \pi$, is the point A satisfies the equation 2 tan x + x = 0. (5 marks) 8 f(x) = e0.5x - x2, $x \in a$ Find f 9(x). (3 marks) b By evaluating f 9(6) and f 9(7), show that the curve with equation y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(6) and f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(6) and f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(6) and f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2
marks) b By evaluating f 9(6) and f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p, where 6 . (2 marks) b By evaluating f 9(7), show that the curve with equation <math>y = f(x) has a stationary point at x = p. Show that $f 0(x) = 8e^2x \cos 2x$. (4 marks) c Hence, or otherwise, determine which turning point is a maximum and which is a minimum. (3 marks) d Find the points of inflection of f(x). (2 marks) d Find the point where the curve intercepts the y-axis. Give your answer in the form ax + by + c = 0 where a, b and c are integers to be found. (5 marks) E/P 11 The curve C has equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the normal to C at Q. E/P E/P (4 marks) The point Q on C has x-coordinate 1. b Find an equation for the norma curve has equation $f(x) = (x_3 - 2x)e - x$. a Find $f(x) = x(1 + x) \ln x$, x > 0 (6 marks) y The point A is the minimum point of the curve with equation y = f(x) where $f(x) = x(1 + x) \ln x$, x > 0 (6 marks) y The point A is the minimum point of the curve with equation y = f(x) + 2x + 4. curve. a Find f 9(x). y = f(x) (4 marks) b Hence show that the x-coordinate of A is 1+x the solution to the equation x = e - 1 + 2x E/P 16 The curve C is given by the equation x = 4t - 3, 8y = 2, t t > 0 (4 marks) b Hence find an equation of l. (3 marks) Differentiation E/P E/P 17 The curve C is given by the equations x = 2t, y = t2, where t is a parameter. Find an equation of the tangent to C at A (1, 1). E/P (7 marks) 19 A curve C is given by the equations $x = 2 \cos t + \sin 2t$, $y = \cos t - 2 \sin 2t$, 0 0, y > 0, by taking logarithms show that dy dx E/P (6 marks) = $xx(1 + \ln x) 35$ a Given that x = ekx, where a and k are constants, a > 0 and $x \in prove$ that $k = \ln a$. b Hence, using the derivative of ekx, prove that when $y = (2 \text{ marks}) 2x \, dy$ (4 marks) = $2x \ln 2 \, dx$ c Hence deduce that the gradient of the curve with equation y = 2x at the point (2, 4) is ln 16. (3 marks) E/P 36 A population P is growing at the rate of 9% each year and at time t years may be approximated by the formula P = P0(1.09)t, t > 0 where P is regarded as a continuous function of t and P0 is the population at time t = 0. a Find an expression for t in terms of P and P0. (2 marks) b Find the time T years when the population has doubled from its value at t = 0, giving your answer to 3 significant figures. (4 marks) dt E/P 37 Given that $y = (arcsin x)^2$ show that $(1 - E/P dy - x)^2 = 0 dx$ (8 marks) 38 Given that y = x - 2arctan x prove that dy $2 dx^2 E/P d2y 2x) 2 dx = 2x(1 - dx) dy 2x 39$ Differentiate arcsin $\sqrt{1 + x2}$ (8 marks) (8 marks) 269 Chapter 9 E/P 40 A curve C has equation y = ln (sin x), E/P 0 0 f(1.5) = -0.5 < 0 f(1.7) = 0.352 > 0 There is a change of sign between 1.1 and 1.3, between 1.3 and 1.5 and between 1.7, so there are at least three roots in the interval 1.1 < x < 1.6. Calculate the values of f(1.1), f(1.3), f(1.5) and f(1.7). Comment on the sign of each answer. f(x) changes sign at least three times in the interval 1.1 < x < 1.7 so f(x) must equal zero at least three times within this interval. 3 1 a Using the same axes, sketch the graphs of $y = \ln x$ and y = -x. Explain how your diagram shows 1 that the function $f(x) = \ln x - -x$ has only one root. b Show that this root lies in the interval 1.7 < x < 1.8. c Given that the root of f(x) is α , show that $\alpha = 1.763$ correct to 3 decimal places. a $y = 1 \times y = \ln x$ 2 - 1 O 1 2 3 x 1 1 $\ln x - x = 0 \Rightarrow \ln x = x 1$ The equation $\ln x = x 1$ The equation $\ln x = x 275$ Chapter f(x) has a root where f(x) = 0. The curves meet at only one point, so there is only 1 one value of x that satisfies the equation $\ln x = x 275$ Chapter 10 y Online 1 __x Locate the root of 1 $f(x) = \ln x - x + 1$ $f(1.7) = \ln x - x + 1$ $f(1.7) = \ln 1.7 - x + 1$ $f(1.7) = \ln$ there is a change of sign in your conclusion. c f(1.7625) = -0.00064... < 0 f(1.7635) = 0.00024... > 0 There is a change of sign in the interval (1.7635) so $\alpha = 1.7635$, so $\alpha = 1.76355$, so $\alpha = 1.76355$, so $\alpha = 1.763555$, so $\alpha = 1.763$ to the given value. Numbers in this range will round up to 1.763 to 3 d.p. 1.762 1.7625 1.763 1.764 x Exercise 10A 1 Show that each of these functions has at least one root in the given interval. a $f(x) = x^2 - \sqrt{x} - 10$, 3 < x < 4 d $f(x) = ex - \ln x - 5$, 1.65 < x < 1.75 $2 f(x) = 3 + x^2 - x^3$ a Show that the equation f(x) = 0 has a root, α , in the interval [1.8, 1.9]. (2 marks) b By considering a change of sign of f(x) in a suitable interval, verify that $\alpha = 1.864$ correct to 3 decimal places. (3 marks) E $3 h(x) = 3\sqrt{x} - \cos x - 1$, where x is in radians. a Show that the equation h(x) = 0 has a root, α , between x = 1.4 and x =1.5. (2 marks) b By choosing a suitable interval, show that $\alpha = 1.441$ is correct to 3 decimal places. (3 marks) E 4 f(x) = sin x - ln x, x > 0, where x is in radians. a Show that f(x) = 0 has a root, α , in the interval [2.2, 2.3]. (2 marks) b By considering a change of sign of f(x) in a suitable interval, verify that $\alpha = 2.219$ correct to 3 decimal places. (3 marks) E 4 f(x) = sin x - ln x, x > 0, where x is in radians. P 5 f(x) = 2 + tan x, 0 < x < π , where x is in radians. a Show that f(x) changes sign in the interval [1.5, 1.6]. b State with a reason whether f(x) has a root in the interval [1.5, 1.6]. 276 Numerical methods P 1 6 A student writes: y = f(x) has a vertical asymptote within this interval so even though there is a change of sign, f(x) has no roots in this interval. By means of a sketch, or otherwise, explain why the student is incorrect. 7 $f(x) = (105x3 - 128x2 + 49x - 6) \cos 2x$, where x is in radians. The diagram shows a sketch of y = f(x). a Calculate f(0.2) and f(0.3). b Use your answer to part a to make a conclusion about the number of roots of f(x) in the interval 0.2 < x < 0.8. c Further calculate f(0.3), f(0.4), f(0.5), f(0.6) and f(0.7). d Use your answers to parts a and c to make an improved conclusion about the number of roots of f(x) in the interval 0.2 < x < 0.8. y 0.5 O - 0.5 P 8 a Using the same axes, sketch the graphs of y = e-x and y = x2. b Explain why the function $f(x) = e - x - x^2$ has only one root. c Show that the function $f(x) = e - x - x^2$ has a root between x = 0.70 and x = 0.71. P 9 a On the same axes, sketch the graphs of $y = \ln x$ and y = ex - 4. b Write down the number of roots of the equation $\ln x = ex - 4$. b Write down the number of x = 0.70 and x = 0.71. P 9 a On the same axes, sketch the graphs of $y = \ln x$ and y = ex - 4. b Write down the number of roots of the equation $\ln x = ex - 4$. b Write down the number of x = 0.70 and x = 0.70. $E/P \ 1 \ x \ y = f(x) \ 10 \ h(x) = \sin 2x + e^4x \ a$ Show that there is a stationary point, α , of y = h(x) in the interval, verify that $\alpha = -0.823$ correct to 3 decimal places. (2 marks) $E/P \ 2 \ 11 \ a$ On the same axes, sketch the graphs of $y = \sqrt{x}$ and $y = x \ (2 \ marks) \ 2 \ b$ With reference to your sketch, explain why the equation $\sqrt{x} = x$ has exactly one real root. (1 mark) 2 c Given that $f(x) = \sqrt{x} - x$, show that the equation $\sqrt{x} = x$ may be written in the form x = q, where p and q are integers to be found. (2 marks) 2 (1 mark) 2 (1 m c), find the values of the constants a, b and c. (3 marks) (3 marks) 277 Chapter 10 10.2 Iterative method, rearrange f(x) = 0. To perform an iterative method, rearrange f(x) = 0 by an iterative method, rearrange f(x) = 0. 0 into the form x = g(x) and use the iterations get closer and closer to the root from the same direction. Graphically these iterations create a series of steps. The resulting diagram is sometimes referred to as a staircase $f(x) = x^2 - x - 1$ can produce the
iterative formula $xn + 1 = \sqrt{xn + 1}$ when f(x) = 0. Let x0 = 0.5. Successive iterations produce the following staircase diagram. y = x + 1 to find x1. You can read across to the line y = x to 'map' this value back onto the x-axis. Repeating the diagram process shows the values of xn converging to the root of f(x). y = x + 1, which is also the root of f(x). y = x + 1 of x +cobweb diagram. Watch out By rearranging the same function $f(x) = x^2 - x - 1$ can produce the iterative formula in different ways you can find different ways you can find different iterative 1 formulae, which may converge differently. xn + 1 = when f(x) = 0. Let x0 = -2. xn - 1 Successive iterations produce the cobweb diagram, shown on . the right. Not all iterations or starting values converge to a root. When an iteration moves away from a root, often increasingly quickly, you say that it diverges. $y = x^2 - 1$ $y = x^2 - 1$ when f(x) = 0. Let $x^0 = 2$. y x0 x Numerical methods Example 4 $f(x) = x^2 - 4x + 1$ a Show that the equation f(x) = 0 can be written as x = 4 - x, $x \neq 0$. f(x) has a root, α , in the interval 3 < x < 4. 1 b Use the iterative formula xn + 1 = 4 - x, $x \neq 0$. f(x) has a root, α , in the interval 3 < x < 4. 1 b Use the iterative formula xn + 1 = 4 - x, $x \neq 0$. f(x) has a root, α , in the interval 3 < x < 4. 1 b Use the iterative formula xn + 1 = 4 - x, $x \neq 0$. f(x) = 0 and $x^2 - 4x + 1 = 0$. subtract 1 from each side. Divide each term by x. This step is only valid if $x \neq 0.1$ b x1 = 4 - x = 3.6666666... 0 1 x2 = 4 - x = 3.72727... Online Use the iterative formula to work out x1, x2 and x 3. You can use your calculator to find each value quickly. 1 x3 = 4 - x = 3.73170... Example 5 f(x) = x3 - 3x2 - 2x + 5 a Show that the equation xn3 - 2xn + 5to calculate the values of x1, x2 and x3, 3 giving your answers to 4 decimal places and taking: ii $x^0 = 4$ i $x^0 = 1.5$ a $f(3) = (3)^3 - 3(3)^2 - 2(3) + 5 = -1$ $f(4) = (4)^3 - 3(4)^2 - 2(4) + 5 = 13$ There is a change of sign in the interval 3 < -1f(x) = 0 has a root in the interval 3 < x < 4, b Use the iterative formula $xn + 1 = \sqrt{2}$ The graph crosses the x-axis between x = 3 and x = 4. 3 = 1.2544...f(x) in this interval. $\sqrt{x} = \sqrt{x} = \sqrt{x03} - 2x0 + 5$ b i x1 = = 1.3385... 3 x2 1 - 2x1 + 5 x32 - 2x2 + 533 = 1.2200... Each iteration gets closer to a root, so the x < 4, and f is continuous, so there is a root of x2 1 - 2x1 + 5 $\sqrt{x} = \sqrt{x} = \sqrt{y}$ Online x03 - 2x0 + 5 ii x1 = sequence x0, x1, x2, x3,... is convergent. 279 Chapter 10 = 4.5092... 3 x Explore the iterations graphically using technology. 33 = 5.4058...x3 Exercise 2 - 2x2 + 53 3 Each iteration gets further from a root, so the $x^2 + 22$ iii $x = 6 - ii x = \sqrt{6x - 2} i x =$ sequence x0, x1, x2, x3,... is divergent. = 7.1219... 10B P 1 f(x) = x2 - 6x + 2 a Show that f(x) = 0 can be written as: x 6 b Starting with x0 = 4, use each iterative formula to find a root of the equation f(x) = 0. Round your answers to 3 decimal places. c Use the quadratic formula to find the roots to the equation f(x) = 0, leaving your answer in the form $a \pm \sqrt{b}$, where a and b are constants to be found. P 2 $f(x) = x^2 - 5x - 3a$ Show that f(x) = 0 can be written as: $x^2 - 3i = x^2 - 3i = x^2 - 3i$ i x = $\sqrt{5x + 35}$ b Let x0 = 5. Show that each of the following iterative formulae gives different roots of f(x) = 0. $i xn + 1 = \sqrt{5xn + 3} E/P x2n - 3 ii$ $5 \ 3 \ f(x) = x^2 - 6x + 1$ Show that the equation f(x) = 0 can be written as $x = \sqrt{6x - 1}$. (1 mark) $\sqrt{(2 \text{ marks})}$ Sketch on the same axes the graphs of y = x and y = 6x - 1. Write down the number of roots of f(x). (1 mark) Use your diagram to explain why the iterative formula xn + 1 = $\sqrt{6xn - 1}$ converges to (1 mark) xn + 1 =a root of f(x) when x0 = 2. $2+1 \ge 0$ can also be rearranged to form the iterative formula xn + 1 = 0_ 6 e By sketching a diagram, explain why the iteration diverges when x0 = 10. (2 marks) a b c d P 4 f(x) = xe-x - x + 2 | | x a Show that the equation f(x) = 0 can be written as x = ln _____, x \neq 2. x-2 f(x) has a root, α, in the interval -2 < x < -1. xn b Use the iterative formula xn + 1 = ln $x \neq 2$ with $x_0 = -1$ to find, to 2 decimal places, $x_n - 2$ the values of x_1 , x_2 and x_3 . | 280 | Numerical methods P 5 f(x) = $x_3 + 5x_2 - 2$ a Show that f(x) = 0 can be written as: 3 is $x = \sqrt{2 - 5x_2} \sqrt{2}$ ii x = 2 - 5x iii x = -5 b Starting with $x_0 = 10$, use the iterative formula in part a (i) to find a root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. Round your answer to 3 decimal places. c Starting with x0 = 1, use the iterative formula in part a (iii) to find a different root of the equation f(x) = 0. a Show that the equation f(x) = 0 can be written as $x = \sqrt{px4} + q$, where p and q are constants to be found. (2 marks) 3 b Let x0 = 0. Use the iterative formula $xn + 1 = \sqrt{pxn4 + q}$, together with your values of (3 marks) p and q from part a, to find, to 3 decimal places, the values of x1, x2 and x3. 3 The root of f(x) = 0 is α . c By choosing a suitable interval, prove that $\alpha = -1.132$ to 3 decimal places. E/P 7 f(x) = 3 cos (x2) + x - 2 a Show that the equation f(x) = 0 can be written as x = (arccos (3)) 2-x (3 marks) 12 (2 marks) 2 - xn b Use the iterative formula xn + 1 = (arccos (3)) 2-x (3 marks) 12 (2 marks) 12 c Given that f(x) = 0 has only one root, α , show that $\alpha = 1.1298$ correct to 4 decimal places. (3 marks) E/P 8 f(x) = 4 cot x - 8x + 3, 0 < x < π , where x is in radians. a Show that the equation f(x) = 0 can be written in the form x = 1.298 correct to 4 decimal places. (3 marks) E/P 8 f(x) = 4 cot x - 8x + 3, 0 < x < π , where x is in radians. a Show that there is a root α of f(x) = 0 in the interval [0.8, 0.9]. cos x 3 b Show that the equation f(x) = 0 can be written in the form x = 1.298 correct to 4 decimal places. x0 = 0.85 to calculate the values of 2 sin xn 8 x1, x2 and x3 giving your answers to 4 decimal places. (2 marks) (3 marks) -2x) + 1, x < 2 (2 marks) 281 Chapter 10 The root of g(x) = 0 is α . The iterative formula xn + 1 = ln (15 - 2xn) + 1, x0 = 3, is used to find a value for α . b Calculate the values of x1, x2 and x3 to 4 decimal places. (3 marks) c By choosing a suitable interval, show that α = 3.16 correct to 2 decimal places. E/P 10 The diagram shows a sketch of part of the curve with equation y = f(x), where f(x) = xex - 4x. The curve cuts the x-axis at the points A and B and has a minimum turning point at P, as shown in the diagram. (3 marks) b Find f(x) = xex - 4x. The curve cuts the x-axis at the points A and the coordinates of A and the coordinates of B. (3 marks) b Find f(x) = xex - 4x. The curve cuts the x-axis at the points A and B and has a minimum turning point at P. as shown in the diagram. (3 marks) b Find f(x) = xex - 4x. Show that the x-coordinate of P is the solution to the equation $x = \ln (x_1 + 1)$ is used. e Let $x_0 = 0$. Find the values of x_1, x_2, x_3 and x_4 . Give your answers to 3 decimal places. (3 marks) 10.3 The Newton-Raphson method The Newton-Raphson method can be used to find numerical solutions to equations of the form f(x) = 0. You need to be able to differentiate f(x) to use this method. The Newton-Raphson procedure. f(xn) xn + 1 = xn $f(x_1)$ The method uses tangent lines to find increasingly accurate approximations of a root. The value of $x_1 + 1$ is the point at which the tangent line at point $(x_1, f(x_1))$ root O 282 x2 x1 x0 x Numerical methods If the starting value is not chosen carefully, the Newton-Raphson method can converge on a root very slowly, or can fail completely. If the initial value, x0, is close to zero, then the tangent at (x0, f(x0)) will intercept the x-axis a long way from x0. y y = f(x) x0 O x1 x Because x0 is close to a turning point the gradient of the tangent at (x0, f(x0)) is small, so it intercepts the x-axis a long way from x0. (x0, f(x0)) If any value, xi, in the Newton-Raphson method is at a turning point, the method will fail because f9(xi) = 0 and the formula would result in division by zero, which is not valid. Graphically, the tangent line will run parallel to the x-axis a long way from x0. (x0, f(x0)) If any value, xi, in the Newton-Raphson method is at a turning point, the method will fail because f9(xi) = 0 and the formula would result in division by zero, which is not valid. axis, therefore never intersecting. y = f(x) x O tangent line will never intersect x-axis Example 6 The diagram shows part of the curve with equation y = f(x), where $f(x) = x^2 - 5x - 4$. y The point A, with x-coordinate p, is a stationary point on the curve. The equation f(x) = 0 has a root,
α , in the interval $1.8 < \alpha < 1.9$. y = f(x) p O x a Explainwhy x0 = p is not suitable to use as a first A approximation to α , apply the Newton-Raphson method to f(x) to find a second approximation to α , giving your answer to 3 decimal places. c By considering the change of sign in f(x) over an appropriate interval, show that your answer to part b is accurate to 3 decimal places. 283 Chapter 10 a It's a turning point, so f9(p) = 0, and you cannot divide by zero in the NewtonRaphson formula. b $f9(x) = 3x^2 + 4x - 5$ Using $x^0 = 2 f(x^0) x^1 = x^0 - 10 x^0$ $f9(x0) 2 x1 = 2 - ___ 15 . x1 = 1.86 f(x1) x2 = x1 - ____$ $f9(x1) \cdot 0.139 8517 x2 = 1.86 12.919\ 992\ d$ Use (axn) = anxn - 1 dx Use the Newton-Raphson process twice. Substitute x1 = 1.856 to three decimal places c f(1.8555) = -0.00348 < 0, f(1.8565) = 0.00928 < 0. Sign change in interval [1.8555, 1.8565] therefore x = 1.856 is accurate to 3 decimal places. Exercise E E/I Use a spread sheet package to find successive Newton-Raphson approximations. y Online x Explore how the Newton- Raphson method works graphically using technology. $10C \ 1 \ f(x) = x^3 - 2x - 1$ a Show that the equation f(x) = 0 has a root, α , in the interval $1 < \alpha < 2$. b Using $x^0 = 1.5$ as a first approximation to α , apply the Newton-Raphson procedure once to f(x) to find a second approximation to α , giving your answer to 3 decimal places. $4 \ 2 \ f(x) = x - x + 6x - 10$, $x \neq 0$. a Use differentiation to (α, α) of the equation f(x) = 0 lies in the interval [-0.4, -0.3]. b Taking -0.4 as a first approximation to α , apply the Newton-Raphson process. once to f(x) to obtain a second approximation to α . Give your answer to 3 decimal places. (4 marks) 3 The diagram shows part of the curve with equation 1 3 - 2, x > 0. y = f(x), where $f(x) = x + \sqrt{x}$ The point A, with x-coordinate q, is a stationary point on the curve. The equation f(x) = 0 has a root α in the interval [1.2, 1.3]. a Explain why $x_0 = q$ is not suitable to use as a first approximation to α . Give your answer to 3 decimal places. 284 y $y = f(x) \times OA$ (4 marks) Numerical methods E E 4 f(x) = 1 $-x - \cos(x^2)$ a Show that the equation f(x) = 0 has a root α in the interval 1.4 < α < 1.5. (1 mark) b Using $x^0 = 1.4$ as a first approximation to α , giving your answer to 3 decimal places. (4 marks) c By considering the change of sign of f(x) over an appropriate interval, show that your answer to part b is correct to 3 decimal places. (2 marks) 3 5 $f(x) = x^2 - 2$, x > 0 x a Show that a root α of the equation f(x) = 0 lies in the interval [1.3, 1.4]. (1 mark) b Differentiate f(x) to obtain a second where f(x) = x2 sin x - 2x + 1. The points P, Q, and R are roots of the equation. The points A and B are stationary points, with x-coordinates a and b respectively. a Show that the curve has a root in each mark) (1 mark) b Explain why x0 = a is not suitable to use as a first approximation, apply the Newton-Raphson method to f(x). (1 mark) c Using x0 = 2.4 as a first approximation, apply the Newton-Raphson method to f(x). (1 mark) c Using x0 = 2.4 as a first approximation, apply the Newton-Raphson method to f(x). + 10, x > 3 4 a Show that f(x) = 0 has a root α in the interval [3.4, 3.5]. (2 marks) b Find f9(x). (2 marks) c Taking 3.4 as a first approximation for α , giving your answer to 3 decimal places. (3 marks) Challenge f(x) = 15 + xe - x2 y The diagram shows a sketch of the curve y = f(x). The curve has a horizontal asymptote at $y = _15$. a Prove that the Newton-Raphson method will fail to converge on a 1 root of f(x) = 0 that lies in the interval [-1, 0], giving your answer to 3 d.p. 285 Chapter 10 10.4 Applications to modelling You can use the techniques from this chapter to find solutions. Example 7 The price of a car in \pounds s, x years after purchase, is modelled by the function $f(x) = 15\ 000\ (0.85)x - 1000\ \sin x$, x > 0 a Use the model to find the value, to the nearest hundred \pounds s, of the car 10 years after purchase. b Show that f(x) has a root between 19 and 20. c Find f9(x). d Taking 19.5 as a first approximation, apply the Newton-Raphson method once to f(x) to obtain a second approximation for the time when the value of the car as it gets older. a $f(10) = 15\ 000\ (0.85)10 - 1000\ \sin 10 = 3497.13...$ After 10 years the value of the car is £3500 to the nearest £100. b $f(19) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19 and 20, so there is at least one root in the interval 19 < x < 20. c $f9(x) = 15\ 000\ (0.85)19 - 1000\ \sin 19 = -331.55... < 0$ There is a change of sign between 19\ \sin 10 = -331.55... $(15\ 000)(0.85)x(\ln 0.85) - 1000\cos x d f(19.5) = 15\ 000\ (0.85)19.5 - 1000\sin 19.5 = 25.0693...\ f9(19.5) = (15\ 000)(0.85)19.5(\ln 0.85) - 1000\cos 19.5 = -898.3009...\ f(x)\ xn + 1 = xn - f9(x)\ 25.0693... = 19.5 -$ -898.3009... = 19.528 e In reality, the car can never have a negative value so this model is not reasonable for cars that are approximately 20 or more years old. 286 Substitute x = 10 into the f(x). Unless otherwise stated, assume that angles are measured in radians. Substitute x = 19 and x = 20 into f(x) is continuous, so f(x) must equal zero within this interval. d Use the fact that (ax) = ax ln a. dx Substitute x = 19.50into f(x) and f9(x). Apply the Newton-Raphson method once to obtain an improved second estimate. Numerical methods Exercise P 10D 1 An astronomer is studying the motion of a planet would have moved if it had been travelling on a circular path, M radians: $M = E - 0.1 \sin E$, E > 0 In order to predict the position of the planet at a particular time, the astronomer needs to find π the value of E is a root of the function $f(x) = x - 0.1 \sin x - k$ where k is a constant to be determined. b Taking 0.6 as a first approximation, apply the Newton-Raphson procedure once to f(x) to n obtain a second approximation for the value of E when M = ____ 6 c By considering a change of sign on a suitable interval of f(x), show that your answer to part b is correct to 3 decimal places. P 2 The diagram shows a sketch of part of the curve with equation v = f(t), where f(t) = $(10 - 2(t + 1)) \ln (t + 1) \cdot v$ 1 P The function models the velocity in m/s of a skier travelling in a straight line. v = f(t) a Find the coordinate of P lies between 5.8 and 5.9. O B A t d Show that the t-coordinate of P is the solution to 20 t = $-11 + \ln(\tan + 1)$ e Let t0 = 5. Find the values of t1, t2 and t3. Give your answers to 3 decimal places. P 3 The depth of a stream is modelled by the function y d(x) = $e - 0.6x(x^2 - 3x)$, 0 < x < 3 where x is the distance in metres from $1 + \ln(t + 1)$ An approximation for the t-coordinate of P is found using the iterative formula 20 th + 1 =the left bank of the stream and d(x) is the depth of the stream in metres. x O y = d(x) The diagram shows a sketch of y = -5e-0.6x (ax2 + bx + c), where a, b and c are constants to be found. 1 c Show that d9(x) = -5e-0.6x (ax2 + bx + c), where a, b and c are constants to be found. 1 c Show that d9(x) = -5e-0.6x (ax2 + bx + c), where a, b and c are constants to be found. 1 c Show that d9(x) = -5e-0.6x (ax2 + bx + c), where a b and c are constants to be found. 1 c Show that d9(x) = -5e-0.6x (ax2 + bx + c). 3 19x – 15 iii x = 3x d Let x0 = 1. Show that only one of the three iterations converges to a stationary point of y = d(x), and find the x-coordinate at this point correct to 3 decimal places. E/P y 4 Ed throws a ball for his dog. The vertical $\mathbf{x} =$ t t t = 18 + 80 sin() - 18 cos() 10 10 (3 marks) To find an approximation for the t-coordinate of A, the iterative height of the ball is modelled by the function t t h(t) = 40 sin () $-9 \cos($) -0.5t2 + 9, t > 0 10 10 y = h(t) is shown in the diagram. y = h(t) a Show that the t-coordinate of A is the solution to $\sqrt{A t O}$ tn + 1 = 18 + 80 sin (10) - 18 cos (10) is used. b Let t0 = 8. Find the values of t1, t2, t3 and t4. Give your answers to 3 decimal places. (3 marks) c Find h9(t). (2 marks) d Taking 8 as a first approximation, apply the Newton-Raphson method once to h(t) to obtain a second approximation for the times formula v tn tn when the height of the ball is zero. Give your answer to 3 decimal places. (3 marks) e Hence suggest an improvement to the range of validity of the model. E/P 5 The annual number of non-violent crimes, in thousands, in a large town x years after the year 2000 is modelled by the function x x c(x) = 5e-x + 4 sin (_) + ____, 0 < x < 10 2 2 The diagram shows the graph of y =
c(x). a Find c9(x). (2 marks) b Show that the roots of the following equations correspond to the turning points on the graph of y = c(x). 5 1 i $x = 2 \arccos(e-x-2) 2 4 10$ ii $x = \ln 2$ x $(4 \cos (2) + 1) 288 (2 \text{ marks}) \text{ y } 6 \text{ y} = c(x) 5 4 3 2 1 \text{ O } 2 4 6 8 10 \text{ x} (2 \text{ marks}) (2 \text{ marks}) \text{ Numerical methods } 5 1 \text{ c Let } x0 = 3 \text{ and}$ xn + 1 = 2 arccos(_ e -xn - _). Find the values of x1, x2, x3 and x4. Give your 2 4 answers to 3 decimal places. (3 marks) 10. Find the values of x1, x2, x3 and x4. Give your d Let x0 = 1 and xn + 1 = ln _____ xn $(4 \cos(2) + 1)$ answers to 3 decimal places. (3 marks) A councillor states that the number of non-violent crimes in the town was increasing between October 2000 and June 2003. e State, with reasons whether the model supports this claim. (2 marks) Mixed exercise 10 E/P 1 $f(x) = x^3 - 6x - 2\sqrt{2}$ b a Show that the equation f(x) = 0 can be written in the form $x = \pm a + x$, and state the values of the integers a and b. (2 marks) f(x) = 0 has one positive root, α . $\sqrt{2}$ iterative formula xn + 1 = a + x, x0 = 2 is used to find an approximate value for α . n (3 marks) b Calculate the values of x1, x2, x3 and x4 to 4 decimal places. (3 marks) E/P 1 2 f(x) = + 3 4-x a Calculate f(3.9) and f(4.1). (2 marks) b Explain why the equation f(x) = 0 does not have a root in the interval 3.9 < x < 4.1. (2 marks) The equation f(x) = 0 has a single root, α . c Use algebra to find the exact value of α . E/P (2 marks) b State the number of positive roots and the number of negative roots of the equation $x^2 + ex - 4 = 0$. (1 mark) 1 c Show that the equation $x^2 + ex - 4 = 0$ can be written in the form $x = \pm(4 - ex)^2$ (2 marks) 1 2 The iterative formula $xn + 1 = -(4 - exn)^2$, $x^2 = -2$, is used to find an approximate value for the negative root. d Calculate the values of x^1 , x^2 , x^3 and x^4 to 4 decimal places. (3 marks) e Explain why the starting value x0 = 1.4 will not produce a valid result with this formula. (2 marks) 289 Chapter 10 E/P 4 g(x) = x5 - 5x - 6 a Show that the equation g(x) = 0 can be written as x = (px + q), where p, q and r are integers to be found. (2 marks) r The iterative formula xn + 1 = (px + q)r $x_0 = 1$ is used to find an approximate value for α . 1 c Calculate the values of x1, x2 and x3 to 4 decimal places. (3 marks) d By choosing a suitable interval, show that $\alpha = 1.708$ is correct to 3 decimal places. (3 marks) d By choosing a suitable interval, show that $\alpha = 1.708$ is correct to 3 decimal places. (3 marks) d By choosing a suitable interval, show that $\alpha = 1.708$ is correct to 3 decimal places. (3 marks) d By choosing a suitable interval, show that $\alpha = 1.708$ is correct to 3 decimal places. (3 marks) d By choosing a suitable interval. same axes the graphs of y = x and $y = \sqrt{3x + 5}$. (2 marks) c Use your diagram to explain why the iterative formula $xn + 1 = \sqrt{3xn + 5}$ converges (1 mark) to a root of g(x) when x0 = 1. $2-5 \times n g(x) = 0$ can also be rearranged to form the iterative formula $xn + 1 = \sqrt{3} d$ With reference to a diagram, explain why this iterative formula diverges when x0 = 7. (3 marks) E/P 6 f(x) = 5x - 4 sin x - 2, where x is in radians. a Show that f(x) = 0 has a root, α , between x = 1.1 and x = 1.15. b Show that f(x) = 0 can be written as x = p sin x + q, where p and q are rational numbers to be found. (2 marks) (2 ma your values (3 marks) of p and q to calculate the values of x1, x2, x3 and x4 to 3 decimal places. E/P 1 7 a On the same axes, sketch the graphs of y = x + 3. (2 marks) 1 b Write down the number of roots of the equation x = x + 3. (2 marks) 1 b Write down the number of x = x + 3 lies in the interval (0.30, 0.31). 1 2 d Show that the equation x = x + 3 may be written in the form x + 3x - 1 = 0. (2 marks) (2 marks) e Use the quadratic formula to find the positive root of the equation q(x) = 0 lies in the interval [6.5, 6.7]. (2 marks) b Taking 6.6 as a first approximation to α , apply the Newton-Raphson process once to g(x) to obtain a second approximation to α . Give your answer to 3 decimal places. (4 marks) d Calculate the percentage error of your answer to 3 decimal places. (4 marks) $\pi \pi 9 f(x)$ = 2 sec x + 2x - 3, - _ < x < _ where x is in radians. 2 2 a Show that f(x) = 0 has a solution, α , in the interval 0.4 < x < 0.5. (2 marks) b Taking 0.4 as a first approximation to α , apply the Newton-Raphson process once to f(x) to obtain a second approximation to α . Give your answer to 3 decimal places. (4 marks) c Show that x = -1.190 is a different solution, β , of f(x) = 0 correct to 3 decimal places. (2 marks) E/P 3 1 10 f(x) = e0.8x - _____, x \neq _____ 3 - 2x 2 (3 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the equation f(x) = 0 correct to 3 decimal places. (2 marks) a Show that the eq 0.5e-0.8x c Show that the equation f(x) = 0 can be written in the form $x = p \ln (3 - 2x)$, stating the value of p found in part c to obtain x1, x2 and x3. Hence write down a second root of f(x) = 0 correct to 2 decimal places. (2 marks) E/P dy 11 a By writing y = xx in the form ln y = x ln x, show that $x = x (\ln x + 1)$. dx x b Show that the function f(x) = x - 2 has a root, α , in the interval [1.4, 1.6]. (4 marks) d By considering a change of sign of f(x) over a suitable interval, show that the curve has a root in the interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval, show that the curve has a root in the interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval, show that the curve has a root in the interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval, show that the curve has a root in the interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval, show that the curve has a root in the interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B considering a change of sign of f(x) over a suitable interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B constants a constants a constant a const Write each coordinate correct to 3 decimal places. (3 marks) O 1 1 c Using the iterative formula xn + 1 =_arccos(_ xn), 4 2 with x0 = 0.4, find the values of x1, x2, x3 and x4. Give your answers to 4 decimal places. (3 marks) y = f(x) A C 1 D 2 x - 1 - 2 B 291 Chapter 10 d Using x0 = 1.7 as a first approximation to the root at D, apply the Newton-Raphson procedure once to f(x) to find a second approximation to the root, giving your answer to 3 decimal places. (4 marks) e By considering the change of sign of f(x) over an appropriate interval, show that the answer to part d is accurate to 3 decimal places. (2 marks) e By considering the change of sign of f(x) over an appropriate interval, show that the answer to part d is accurate to 3 decimal places. (2 marks) e By considering the change of sign of f(x) over an appropriate interval, show that the answer to part d is accurate to 3 decimal places. Points A and B are the points of inflection on the curve. y A O y = $f(x) \times B$ a Show that equation f(x) = 0 can be written as: 7 - 15x i x = $\sqrt{7 - 3x}$ iii x = Explain why you cannot use the same iterative formula to find an approximation for the x-coordinate of A. d Use the Newton-Raphson method to find an estimate for the x-coordinate of A, correct to 3
decimal places. Summary of key points 1 If the function f(x) is continuous on the interval [a, b] and f(a) and f(b) have opposite signs, then f(x) has at least one root, x, which satisfies a < x < b. 2 To solve an equation of the form f(x) = 0 by an iterative method, rearrange f(x) = 0 into the form x = g(x) and use the iterative formula xn + 1 = g(xn). 3 The Newton-Raphson formula for approximating the roots of a function f(x) is f(xn) xn + 1 = xn - 1f9(xn) 292 11 Integration Objectives After completing this chapter you should be able to: \bullet Integrate standard mathematical functions including trigonometric and exponential functions of the chain rule to \rightarrow pages 294-298 integrate functions of the chain rule to \rightarrow pages 294-298 integrate functions of the chain rule to integrate functions of the chain rule functio more complex \rightarrow pages 300-303 functions \bigcirc Integrate functions by making a substitution, using integration by \rightarrow pages 313-317 \bigcirc Use the trapezium rule to approximate the area under a curve. \rightarrow pages 317-322 \bigcirc Solve simple differential equations and model real-life situations \rightarrow pages 322-329 with differential equations Prior knowledge check 1 Differentiate: a (2x - 7) 6 b sin 5x x c e 3 2 \leftarrow Sections 9.1, 9.2, 9.3 1 b find $\int 4 f(x) dx 9 3 1 Given f(x) = 8x 2 - 6x - 2a find <math>\int f(x) dx \leftarrow$ Year 1, Chapter 13 3x + 22 Write as partial fractions. $(4x - 1)(x + 3) \leftarrow$ Section 1.3 4 Find the area of the region R bounded by the curve $y = x^2 + 1$, the x-axis and the lines x = -1 and x = 2. y = 5 4 Integration can be used to solve differential equations. Archaeologists use differential equations to estimate the age of fossilised \rightarrow Exercise 11K Q9 plants and animals. $3 \ge 1 - 2 - 1$ O R $1 \ge x \leftarrow$ Year 1, Chapter 13 293 Chapter 11 11.1 Integrating standard functions Integration is the inverse of differentiation. You can use your knowledge of derivatives to integrate familiar functions. Watch out This is true for all values of n 1 2 3 4 5 6 7 8 9 x $\int x n dx = -\cos x + c \int \cos x dx = \sin x + c \int \sin x dx = -\cos x + c \int \sec 2x dx = \tan x + c \int \csc x dx = -\csc x + c \int \frac{1}{2} x dx = -\cos x + c \int \frac{1}{2$ $cosec x dx = -\cot x + c \int sec x tan x dx = sec x + c n+1$ Example 1 dx it is usual to When finding $\int x$ write the answer as $\ln |x| + c$. The modulus sign removes difficulties that could arise when evaluating the integral for negative values of x. Notation Links dy For example, if y = cos x then = -sin x. dx This means that $\int (-sin x) dx = cos x + c$ and hence $\int \sin x \, dx = -\cos x + c$. b $\cos x - 2ex$) $dx \int ($ $\sin 2x \int 2 \cos x \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt{x} \, dx = 3 \ln |x| + c \int \sqrt$ -2ex dx) sin 2 x = $-cosec x - 2ex + c 294 \leftarrow Section 9.1 1$ Find the following integrals. $3 a \int (2 \cos x + x - \sqrt{x}) dx a except -1$. Integrate each term separately. 4. Use 3. Use 1. Use 7 integrals of standard functions and express the integrand in terms of these standard functions and express the integral so don't forget the +c. Look at the list of integrals of standard functions and express the integrand in terms of these standard functions and express the integral so don't forget the +c. Look at the list of integrals of standard functions and express the integral so don't forget the +c. Look at the list of integrals of standard functions and express the integral so don't forget the +c. Look at the list of integrals of standard functions and express the integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integrals of standard functions and express the integrand in terms of these standard functions at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget the +c. Look at the list of integral so don't forget
the +c. Look at the list of i functions. Remember the minus sign. Integration Example 2 Problem-solving Given that a is a positive constant and 3a 2x + 1 fa x dx = ln 12, find the exact value of a. fa 3a 2x + 1 x Integrate as normal and write the limits as a and 3a. Substitute these limits into your integral to get an expression in a and set this equal to ln12. Solve the resulting equation to find the value of a. dx (2 + x) dx = $[2x + \ln x]$ 3a a = $(6a + \ln 3a) - (2a + \ln a) = \int a 3a 1$ Separate the terms by dividing by x, then integrate term by term. 3a = $4a + \ln 3 = \ln 4 + \ln 4 + \ln 3 = \ln 4 + \ln 4 + \ln 4 = \ln 4 + \ln 4 + \ln 4 = \ln 4 + \ln 4 + \ln 4 + \ln 4 = \ln 4 + \ln$ integrated expression. a Use the laws of logarithms: $\ln a - \ln b = \ln () b 12 \ln 12 - \ln 3 = \ln () = \ln 4 3$ Online Use your calculator to check your value of a using numerical integration. 11A 1 Integrate the following with respect to x. 5 2 a 3 sec2 x + + 2 x x c 2(sin x - cos x + x) 2 e 5ex + 4 cos x - 2 x 1 1 1 g + 2 + 3 x x x i 2 cosec x $\cot x - \sec 2x 2 \text{ Find the following integrals. 11 a f (+ 2x) dx \int \sec 2x (1 + \tan 2x) dx \int = \tan 2x (1 + \tan 2x) dx$ $\sin 2x x$) $\int \cos x(1 + \csc 2x) dx \int \sec 2x(1 - \cot 2x) dx 1 + \sin x + \cos 2x \sec x) dx \int (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. Give your answers as exact values. 7 61 + x b } (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\qquad} \cos 2x 3 \text{ Evaluate the following. 6 (\underline{\abla following. 6 (\underline$ functions, always work in radians. 295 Chapter 11 E/P 2a $3x - 1_1 4$ Given that a is a positive constant and $\int a$ $x \, dx = 6 + \ln(2)$, find the exact value of a. E/P 5 Given that a is a positive constant and $\int E/P \, 6$ Given $\int (3ex + 6e - 2x) \, dx = 0$, find the value of b. E/P 4 1 _3 7 f(x) = $_8 x \, 2 - _x$, x > 0 lna ln1 ex + $e - x \, dx = _7$, find the exact value of a. 48 b (4 marks) (4 marks) (4 marks) (4 marks) 2 a Solve the equation f(x) = 0. (2 marks) b Find $\int f(x) dx$, giving your answer in the form $p + q \ln r$, where p, q and r are (2 marks) 4 1 rational numbers. (3 marks) 11.2 Integrating f(ax + b) If you know the integral of a function f(x) you can integrate a function of the form f(ax + b) using the reverse of the chain rule for differentiation. Example 3 Find the following integrals. a $\int \cos(2x + 3) dx = 2 \sin(2x + 3) + c b$ Consider y = e4x + 1 + 4 dx + 1 So $\int e^{4x} + 1 dx = +c 4e c$ Consider y = tan 3x: dy $= sec2 3x \times 3 dx$ $1 \text{ So} \int \sec 2 3x \, dx = 3 \tan 3x + c 296 \, c \int \sec 2 3x \, dx$ Integrating $\cos x$ gives $\sin x$, so try $\sin (2x + 3)$. Use the chain rule. Remember to multiply by the derivative of 2x + 3 which is 2. This is 2 times the required expression so you need to divide $\sin (2x + 3)$ by 2. The integral of ex is ex, so try $e^{4x} + 1$. This is 4 times the required expression so you divide by 4. Recall 6. Let $y = \tan 3x$ and differentiate using the chain rule. This is 3 times the required expression so you divide by 3. Integrate an expression such as $\cos(2x^2 + 3)$ since it is not in the form f(ax + b) + c a Example 4 Find the following $dx 3x + 2b \int (2x + 3)4 dx 1$ Integrating x gives $\ln|x|$ so try $\ln (3x + 2)$. a Consider $y = \ln (3x + 2) dy 1$ So = x 3 dx 3x + 211 So $\int dx = \ln |3x + 2| + c 3x + 23$ The 3 comes from the chain rule. It is 3 times the required expression, so divide by 3. b Consider y = (2x + 3)5 dy So $= 5 \times (2x + 3)4 \times 2 dx = 2$ $10 \times (2x + 3)4$ 1 So $\int (2x + 3)4$ dx = (2x + 3)5 + c 10 To integrate (ax + b)n try (ax + b)n + 1. The 5 comes from the exponent and the 2 comes from the exponent and the exponent and the exponent and the e Hint For part a consider y = cos(2x + 1). You do not need to write out this step once you are confident with using this method. f sec $2x \cot 2x 2$ Find the following integrals. a c e $\int (e^2x - 21 \sin (2x - 1)) dx \int sec^2 2x(1 b d + sin 2x) dx \int (e^3 - x + sin (3 - x) + cos(3 - x)) dx 3$ Integrate the following: 1 a dx 1 2_ sin 2 x 1 b 2(2x + 1) c(2x + 1) 2 3 d 4x - 1 3 f 2(1 - 4x) g(3x + 2) 5 3 h $1 - 4x j \cos 3x - \sin 3x \int (ex + 1)2 dx \int 1 3 - 2 \cos 2x$ $(e5x + (1 - x)5) dx d 1 \int ((3x + 2)2 +$ ____ dx (3x + 2)2) c $\int 5$ Evaluate: a $\int 3\pi$ π 4 4 cos $(\pi - 2x)$ dx b $\int 1$ 12 1 2 dx $(3 - 2x)45\pi$ 18 2 π 9 sec2 $(\pi - 3x)$ dx d 5 $\int 2$ dx 7 – 2x 3 E/P 6 Given $\int 3(2x - 6)2 dx = 36$, find the value of b. (4 marks) E/P e8 1 1 7 Given $\int 2 dx = -7$, find the value of k. e kx 4 (4 marks) E/P b π 8 Given $dx = a \ln(17)$, and that a and b ax + b are integers with 0 < a < 10, find two different pairs of values for a and b. 11.3 3k 4k find the exact value of k. (7 marks) Problem-solving Calculate the value of the indefinite integral in terms of k and solve the resulting equation. Challenge 11 1 1 ____41 Given $\int 5$ $\int \pi (1 - \pi \sin kx) dx = \pi (7 - 6\sqrt{2}),$ Using trigonometric identities and be integrated to be replaced by an identities can be used to integrated. Example Links Make sure you are familiar with the standard trigonometric identities. The list of identities in the summary of Chapter 7 will be useful. \leftarrow page 196 5 Find \int tan2 x dx Since sec2 x; 1 + tan2 x tan2 x; sec2 x - 1 So \int tan2 x dx = \int (sec2 x - 1) dx = f sec2 x dx - \int 1 dx = tan x - x + c 298 You cannot integrate sec2 x directly. Using 6. Integration Example 6 Show that $\int \pi_{-} 8 \pi_{-} 12 \sin 2x \, dx = \pi_{-} 48 + 1 - \sqrt{2}$ You cannot integrate sin2x directly. Use the trigonometric identity to write it in terms of $\cos 2x$. 8 Recall $\cos 2x \equiv 1 - 2 \sin 2x$ 1 So $\sin 2x \equiv (1 - \cos 2x) 2$ $\pi \pi 8 8 1 1$ So $\int \pi \sin 2x \, dx = \int \pi (- \cos 2x) \, dx = 1 - 2 \sin 2x$, dy $= 2 \cos 2x$. Adjust for the constant. dx Substitute the limits into the integrated expression. $\pi = 8 1 1 = 2 \cos 2x$. Adjust for the constant. $\begin{bmatrix} x - \sin 2x \end{bmatrix} \pi \quad 2 \ 4 \ 12 \ \pi \ \pi \ 1 = (-\sin ()) - (-\sin ()) \ 16 \ 4 \ 24 \ 4 \ 6 \ \pi \ \pi \ 1 \qquad \pi \ \pi \ 1 = (-()) - (-()) \ 16 \ 4 \ 24 \ 4 \ 2 \ \sqrt{21}$ $\pi \pi 1 1 \sqrt{2} = (-) + (-) 16 24 4 2 2 1 - \sqrt{2} 3\pi 2\pi = (-) + 48 48 8 1 - \sqrt{2} \pi = +$ 48.8 Watch out This is a 'show that' guestion so don't use you calculator to simplify the fractions. Show each line of your working carefully. 7 Find: a \int sin 3x cos 3x dx a You will save lots of time in your exam if you are familiar with the exact values for trigonometric functions given in radians. π Write sin (__) in its rationalised denominator form, _____4 $\sqrt{2}$ 1___. This will make it easier to as _____ rather than simplify your fractions. Example Problem-solving b $\int (\sec x + \tan x)^2 dx \int \sin 3x \cos 3x dx = \int 21 \sin 6x dx$ $11 = -2 \times 6 \cos 6x + c 1 = -12 \cos 6x + c \log 6x + c \log 2x + 2 \sec x \tan x + (\sec 2x - 1) = 2 \sec 2x + 2 \sec x \tan x - 1 \log \int (\sec x + \tan x)^2 dx = \int (2 \sec 2x + 2 \sec x \tan x - 1) dx = 2 \tan x$ $+ 2 \sec x - x + c$ Remember $\sin 2A \equiv 2 \sin A \cos A$, so $\sin 6x \equiv 2 \sin 3x \cos 3x$. Use the reverse chain rule. 1 Simplify 21×16 to 12 Multiply out the bracket. Write $\tan 2x \operatorname{as sec} 2x - 1$. Then all the terms are standard integrals. Integrate each term using 6 and 9.299 Chapter
11 11C Exercise 1 Integrate the following: For part a, use $1 + \cot 2x \equiv 1$. $\cos^2 x$. For part c, use $\sin^2 A \equiv 2 \sin A \cos A$, making a suitable substitution for A. b $\cos^2 x c \sin^2 x \cos^2 x d (1 + \sin x)^2 e \tan^2 3x f (\cot x - \csc x)^2 g (\sin x + \cos x)^2 h \sin^2 x \cos^2 x d (1 + \sin x)^2 e \tan^2 3x f (\cos x - \sin x)^2 dx 1 + \cos x dx f)$ $\cos 2x f \int (\cot x - \tan x) 2 dx \cos 2x dx \int$ $1 - \cos 2 2x c \int 0 \pi 4 (1 + \sin x)$ $\cos x$)2 $\int (\cos x - \sec x)2 dx h$ 2 2+ π 3 Show that $\int \pi \sin 2x dx =$ 8 4 cos 2x dx \int $2 \cos 2 x \, dx \, d \int \pi$ $2 \, 3\pi$ $8 \, 5 \, a \, By$ expanding sin (3x + 2x) and sin (3x - 2x) using the double-angle formulae, or otherwise, show that sin $5x + \sin x \equiv 2 \sin 3x \cos x$ $1 - \sin 2 x dx$ (4 marks) (3 marks) (3 marks) 6 f(x) = 5 sin 2 x + 7 cos 2 x a Show that f(x) = cos 2x + 6. b Hence, find the exact value of E/P c i b Hence find $\int \sin 3x \cos 2x dx E/P j$ (cos 2x - 1)2 (4 marks) 4 Find the exact value of each of the following: $\pi \pi_{-1} = 1.342 dx b \int a \int \pi_{-1} dx dx = 1.342 dx$ π (sin x – cosec x) dx 2 2 6 sin x cos x 6 E/P 1 i $2 \sin x \cos 2 x b \pi E/P$ Hint a cot $2 x (0 \pi - 4)$ (3 marks) f(x) dx. (4 marks) 7 a Show that cos $4x = 8 \cos 4x + 2 \cos 2x + 8$ (4 marks) 11.3 b Hence find (cos $4x + 2 \cos 2x + 8$ (4 marks) 11.3 b Hence find (cos $4x + 2 \cos 2x + 8$ (4 marks) 11.4 Reverse chain rule f(x) for differentiation. Example 8 $dx 3 + 2 \sin x$ If $f(x) = 3 + 2 \sin x$, then $f'(x) = 2 \cos x$. By adjusting for the constant, the numerator is the derivative of the denominator. Integration a Let I = $2x \int dx 2x + 1$ Consider $y = \ln|x^2 + 1|$ Then This is equal to the original integrand, so you don't need to adjust it. dy 2 x +1 300 b cos x ∫ Problem-solving Find a $2x dx \int$ $\times 2 \cos x 3 + 2 \sin x I = So 1 2 \ln|3 + 2 \sin x| + c \blacksquare$ To integrate expressions of the form $f'(x) \int k dx$, try $\ln|f(x)|$ and differentiate f(x) to check, and then 1 = $\times 2x \, dx \, x^2 + 1 \, I = \ln |x^2 + 1| + c$ So Since integration is the reverse of differentiation. cos x I = \int $dx 3 + 2 \sin x b \text{ Let } y = \ln|3 + 2 \sin x| \text{ Let } dy _ dx 1 = _$ adjust any constant. Try differentiating $y = \ln|3 + 2 \sin x|$. The derivative of $\ln|3 + 2 \sin x|$ is twice the original integrand, so you need to divide it by 2. Watch out You can't use this method to 1 integrate a function such as method with functions of the form kf'(x)(f(x))n. Example 9 Find: a $\int 3 \cos x \sin 2x \, dx$ a Let b $\int x(x^2 + 5)^3 \, dx \, I = \int 3 \cos x \sin 2x \, dx$ Consider $y = \sin 3x \, dy$ and $dx = 4(x^2 + 5)^3 \, dx \, I = \int 3 \cos x \sin 2x \, dx$ Consider $y = \sin 3x \, dy$ and $dx = 4(x^2 + 5)^3 \, dx \, I = \int 3 \cos x \sin 2x \, dx$ Consider $y = \sin 3x \, dy$ and $dx = 4(x^2 + 5)^3 \, dx \, I = \int 3 \cos x \sin 2x \, dx$ Consider $y = \sin 3x \, dy$ and $dx = 4(x^2 + 5)^3 \, dx \, I = \int 3 \cos x \sin 2x \, dx$ Consider $y = \sin 3x \, dy$ and $dx = 4(x^2 + 5)^3 \, dx \, I = \int 3 \cos x \sin 2x \, dx$ Consider $y = \sin 3x \, dy$ and $dx = 4(x^2 + 5)^3 \, dx$. So Try differentiating sin3 x. 1 2 4 I = 8 (x + 5) + c Try differentiating (x2 + 5)4. The 2x comes from differentiating (x2 + 5). This is 8 times the required expression of the form $\int k f'(x)(f(x))n dx$, try (f(x))n + 1 and differentiate to check, and then adjust any constant. 301 Chapter 11 Example 10 cosec2 $3 dx (2 + \cot x)$ This is in the form $\int k f g(x)(f(x)) dx$ with $f(x) = 2 + \cot x$ and n = -3. cosec 2 x Let I = $\int k dx$ x Use integration to find $3 dx (2 + \cot x) Let y = (2 + \cot x) - 2 dy$ ax Use the chain rule. = $-2(2 + \cot x) - 3 \times (-\cos ec 2x)$ This is 2 times the required answer so you need to divide by 2. = $2(2 + \cot x) - 3 \csc 2x - 1 - 2 + c \cos 2x$ $I = 2(2 + \cot x)$ Example 11 θ 15 π Given that $\int 0.5 \tan x \sec 4x \, dx =$ where $0 < \theta <$, find the exact value of θ . 2 4 Let I = This is in the form $\int k f^0(x)(f(x))n \, dx$ with $f(x) = \sec x \, dx \, \theta$ Let $y = \sec 4x \, dx \, \theta$ Let $y = \sec 4x \, dx \, \theta$ Let $y = \sec 4x \, dx \, \theta$ Let $y = \sec 4x \, dx \, \theta$ Let I = This is in the form $\int k f^0(x)(f(x))n \, dx$ with $f(x) = \sec x \, dx \, \theta$ Let $y = \sec 4x \, dx \, dx$ Let $y = \sec 4x \, dx \, dx$ Let $y = \sec 4x \, dx \, dx$ 15 So I = $[sec 4x] = 440(4554) - (45sec 4\theta) = 4sec 4\theta = 44520sec 4\theta = 44520sec 4\theta = 44520sec 4\theta = 44520sec 4\theta = 4252sec 4\theta = 4452sec 4\theta = 4252sec 4\theta = 4452sec 4\theta = 4252sec 4\theta = 4452sec 4\theta = 4252sec 4\theta = 4252sec$ $2x 3 3 + \sin 2x (e + 1) g xe x 2 1 1 sec 0 = _ = _ = 1 cos 0 1 \pi 3\pi 5\pi 1 _ are \theta = -_, _, _, _, _, _... The solutions to cos \theta = \pm _ 4 4 4 4 \sqrt{2 \pi} The only solution within the given range for \theta is _ 4 sec \theta = \pm \sqrt{2 \pi \theta} = _ 4 Exercise Substitute the limits into the integrated expression. h cos 2x (1 + sin 2x)4 Hint x c _ (x2 + 4)3 sin 2x f _ 3 (3 + cos 2x) i sec 2 x tan 2 x Decide carefully whether each expression f9(x) is in the form k _ or f(x) kf9(x)(f(x))n . j sec 2 x (1 + tan 2 x) Integration 2 Find the following integrals. a c e f(x + 1)(x2 + 2x + 3)4 dx f sin 5 3x cos 3x dx e 2x dx f _ e2x + 3 b d f f cos e 2 2x cot 2x dx f cos x esin x dx f x(x2 + 1) dx 3 2 g f (2x + 1)\sqrt{x2 + x + 5}$ $\int \frac{1}{\sqrt{x^2 + x + 5 i \sin x \cos x}} dx \int \frac{1}{\sqrt{y} \sin x \cos x} \int \frac{1}{\sqrt{y} \sin x} \cos x} \int \frac{1}$ dx h 2x + 1dx∫ 99п 6 sin 3x) u 2 x 2 du 2 Rewrite I in terms of u and simplify. 1 1 2 = $\int 4(u - 5)u \, du$ 1 1 32 2 = $\int 4(u - 5u) \, du$ Multiply out the brackets and integrate using rules from your Year 1 book. \leftarrow Year 1, Chapter 13 3 5 5u 2 u 2 -+c = x 5 3 4 4 x 2 2 1 3 = 0 10 - 5u 2 6 Simplify. + c 5 3 $1 \ 1 \ \text{So I} = \int ($ $+ c 6 10 b Let Finally rewrite the answer in terms of x. I = <math>\int x\sqrt{2x + 5} dx$ First find the relationship between dx and du. $u^2 = 2x + 5$ Using implicit differentiation, cancel 2 and rearrange to get dx = udu. du 2u = 2 dx So replace dx with udu. $\overline{(2x + 5)}$ 2 5(2x + 5)2 So I = terms of u. You will need to make x the subject of $u^2 = 2x + 5$. and $u^2 - 5x = 2$ So $u^2 - 5I = f(- -)$ du 2 2 Multiply out the brackets and integrate. $u^5 5u^3 = - + c 6 10 5 (2x + 5) 2 So I = 10 3 5(2x + 5) 2 - 6 + c$ Rewrite answer in terms of x. Example 13 Use the substitution $u = \sin x + 1$ to find $\int \cos x \sin x (1 + \sin x)^3 dx$ Let Let $I = \int \cos x \sin x (1 + \sin x)^3 dx$ u = sin x + 1 du ___ = cos x dx So substitute cos x dx with du. 304 First replace the dx. cos x appears in the integrand, so you can write this as du = cos x dx and substitute. Integration Use u = sin x + 1 to substitute for the remaining terms, rearranging where evaluate a definite integral, you have to be careful of whether your limits are x values or u values. You can use a table to keep track. u=x+1 du dx = 1 305 Chapter 11 so replace (x + 1)3 with u3, and x with u - 1. x 2 0 u 3 1 Chapter 11 so replace (x + 1)3 with u3, and x with u - 1. x 2 0 u 3 1 Chapter 11 so replace dx with du and replace (x + 1)3 with u3, and x with u - 1. x 2 0 u 3 1 Chapter 11 so replace dx with u - 1. x 2 0 u 3 1 Chapter 11 so replace dx with
u - 1. x 2 0 u 3 1 Chapter 11 so replace dx with u - 1. x 2 0 u 3 1 Chapter 11 so r du 3 So I = Note that the new u limits replace their corresponding x limits. u5 u4 = [--] 5 4 1 3 Multiply out and integrate. Remember there is no need for a constant of integrals. 243 81 1 1 = (--) - (--) 5 4 5 4 = 48.4 - 20 = 28.4 b The integral can now be evaluated using the limits for u without having to change back into x. $\int 0 \cos x \sqrt{1 + \sin x} \, dx \pi$ 2 du u = 1 + sin x \Rightarrow = cos x, so replace dx $\cos x \, dx$ with du and replace $\sqrt{1 + \sin x}$ with 1 u 2. x u 2 0 2 π 1 So I = = $\int 1 2$ 1 u 2 du Use u = 1 + sin x. Remember that limits for integrals involving trigonometric functions will always be in radians. $\pi x = 0$, means u = 1 + 1 = 2 and x = 0, means 2 u = 1 + 0 = 1. Rewrite the integral in terms of u. 2 32[3u]1 3 222 = (32) - (3) $2\sqrt{SoI} = 3(22 - 1)c$ e You could also write convert the integral back into a function of x and use the original limits. $11E \int x \sqrt{1 + x} dx$; u = 1 + x b $\int sin 3x dx$; $u = cosx d \int sec 2x tan x \sqrt{1 + tan x} dx$; $u = \frac{1 + tan x}{1 + tan x} dx$; $u = \frac{1 + x}{1 + x}$ f 2 Use the substitutions given to find the exact values of: a c d 306 $\int 0x/x + 4 dx$; u = x + 45 π _____ $\int 0 \sin x/3 \cos x + 1 dx$; u = cos x 2 π _ 3 ___ Problems solution find the substitutions given to find the exact values of: a c d 306 $\int 0x/x + 4 dx$; u = x + 45 π _____ $\int 0 \sin x/3 \cos x + 1 dx$; u = cos x 2 π _ 3 ___ Problems solution find the substitutions given to find the usual way. _____ $\int 0 \sec x \tan x/\sec x + 2 dx$; u = sec x _____ b = 1 + x = 1 $\int_1 x 2\sqrt{1-x} 2 \, dx. \sqrt{3-9} (8 \text{ marks}) (5 \text{ marks})$ dv To use integration by parts you need to write the function you are integrating in the form u dx dv You will have to choose what to set as dx 307 Chapter 11 Example 16 Find $\int x \cos x \, dx \, I = \int x \cos x \, dx \, du \, u = x \Rightarrow = 1 \, dx \, dv = \sin x \, dx$ Using the integration by parts formula: Let I = x sin x - $\int \sin x \times 1 \, dx$ = x sin x + cos x + c Example 17 Problem-solving For expressions like x cos x, x2sin x and x 3ex let u equal the x term. When the expressions for u, v, and dx dx dv Take care to differentiate u but integrate dx du Notice that for dx is a simpler integral than dx dv fu dx. dx Find fx2 ln x dx Let I = fx2 ln x dx du 1 u = ln x \Rightarrow = dx x dv x3 = $x^2 \Rightarrow v^2$ = dx 3 3 3 x x 1 I = $\ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 3 x^3 x^2 = \ln x - \int dx 3 x^3 x^3 = \ln x - \int dx 3 x^3 x^3 x^2 = \ln x - \int dx 3 x^3 x^3 x^2 = \ln x - \int dx 3 x^3 x^3 = \ln x - \int dx 3 x^3 x^3 = \ln x - \int dx dx = \ln x + \ln$ dv Take care to differentiate u but integrate dx Apply the integration by parts formula. du Simplify the v term. dx It is sometimes necessary to use integration by parts twice, as shown in the following example. Example 18 Find $x^2 ex dx Let So I = \int x^2 ex dx du u = x^2 \Rightarrow = 2x dx dv = ex \Rightarrow v = ex dx I = x^2 ex - \int 2xex dx du u = 2x \Rightarrow = 2$ $dx dv = e x \Rightarrow v = ex dx 308 dv$ There is no ln x term, so let $u = x^2$ and $ex^2 = ex^2 + ex^2 +$ use integration by dv parts again with u = 2x and = ex. dx Integration by parts formula for a second time. $= x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex - 2xex + \int 2ex dx = x^2 ex + 2ex + 2e$ be integrated as $\ln x \times 1$, dv then $u = \ln x$ and = 1. dx Remember if an expression involves $\ln x$ you should always set $u = \ln x$. $\ln x \times 1$ dx du $1 u = \ln x \Rightarrow = dx x dv = 1 \Rightarrow v = x dx 2 I = [x \ln x] 1 - \int 1 2 1 x \times x dx = (2 \ln 2) - (1 \ln 1) - 2 \ln 2 - [x] 21 \int 1 2 Problem-solving du Apply limits to the uv term and the <math>\int v dx$ term dx separately. 1 dx = $2 \ln 2 - (2 - 1)$ Evaluate the limits on uv and remember $\ln 1 = 0$. = $2 \ln 2 - 1$ Exercise 11F 1 Find the following integrals. a d fx sin x dx f sec x tan x dx b e fxec x dx x dx f sin 2 x c Hint You will need to use these standard results. In your exam they will be given in the formulae booklet: fx sec 2 x dx • 2 Find the following integrals. integrals. a d $\int 3 \ln x \, dx \int (\ln x) 2 \, dx \, b \, e \ln x \int x \ln x \, dx \, c \int dx \, x^3 \int (x^2 + 1) \ln x \, dx \, 3$ Find the following integrals. a $\int x^2 e^{-x} \, dx \, b \int x^2 \cos x \, dx \, 4$ Evaluate the following: a $e \int 0 \ln 2 x e^{2x} \, dx \, 1 \int 0 \, 4x(1 + x)^3 \, dx \cdot b \, f \pi$ c $\cdot \int 12x^2(3 + 2x)^5 \, dx \int 0 \, x \sin x \, dx \, \pi \int 0 \, x \cos^2 14 \, x \, dx \, 2 \cdot c \, g \, d \, \pi$ $\ln|\sin x| + c \int \csc x \, dx = -\ln|\csc x + \cot x| + c \int 2x \sin 2x \, dx \int 0 x \cos x \, dx \int 0 x \cos x \, dx \int 0 x \cos x \, dx \int 0 \sin x \ln (\sec x) \, dx \, d \int 1 2 \ln x$ (2 marks) b Use your answer to parts to find $\int x \cos 4x \, dx$. (3 marks) b Use your answer to parts to find $\int x \cos 4x \, dx$. (3 marks) b Use your answer to parts to find $\int x \cos 4x \, dx$. (3 marks) b Use your answer to parts to find $\int x \cos 4x \, dx$. (3 marks) b Using integration by parts, or otherwise, show that $\int (x - 2)\sqrt{8} - x \, dx = -25(8 - x)(x + 2) + c$ ______3 (6 marks) 2 c Hence find $\int (x - 2)\sqrt{8} - x \, dx$. _____7 (2 marks) 4 E/P 7 a Find $\int \sec 2 3x \, dx$. (3 marks) b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2
3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x \sec 2 3x \, dx$. [3 marks] b Using integration by parts, or otherwise, find $\int x$ q. (6 marks) 9 (4 marks) 11.7 Partial fractions are be used to integrate algebraic fractions. Using partial fractions as partial fractions. $= (x + 1)(x - 2) x + 1 x - 2 c 2 \int dx 1 - x2$ Split the expression to be integrated into partial fractions. So $x - 5 \equiv A(x - 2) + B(x + 1)$ Let x = 12 dx (x + 1)(x - 2) (2x + 1)(x - 2) x - 5 A B a_____ dx b ∫ _____ Chapter 1 Example 20 Use partial fractions to find the following integrals. $8x^2 - 19x + 1x^{-5}a$ -1: -6 = A(-3) so A = 2 Let x = 2: -3 = B(3) so B = -1 x - 5 So $\int dx (x + 1)(x - 2) 2 \overline{1 = \int (a - 1)^2 dx} dx (x + 1)(x - 2) \overline{1 = \int (a - 1)^2 dx} dx (x + 1)(x - 2) \overline{1 = \int (a - 1)^2 dx} dx (x + 1)(x - 2) \overline{1 = \int (a - 1)^2 dx} dx (x + 1)(x - 2) \overline{1 = \int (a - 1)^2 dx} dx (x + 1)(x - 2) \overline{1 = \int (a - 1)^2 dx} dx (x + 1)(x - 2) \overline{1 = \int (a - 1)^2 dx} dx (x +$ + c x-2 Let x = -1 and 2. Rewrite the integral and integrate each term as in \leftarrow Section 11.2 Remember to use the modulus when using ln in integration. The answer dx (2x + 1)(x - 2)2 2 8x2 - 19x + 1 $(2x + 1)(x - C A B \equiv + 2 + 2x + 1 (x - 2) x - 2 2)$ It is sometimes useful to label the integral as I. Remember the partial fraction form for a repeated factor in the denominator. $8x^2 - 19x + 1 = A(x - 2)^2 + B(2x + 1) + C(2x + 1)(x - 2)$ Let x = 2: -5 = 0 + 5B + 0 so B = -125 1 1 Let x = -2: 122 = 4A + 0 + 0 so A = 2 Let x = 0: Then $1 = \overline{4A + B} - 2C1 = 8 - \overline{1 - 2C}$ so $C = 33I = \int (+ -2)^2 x - 2 \int 1^2 x + 1 (x - 2)^2 x - 2 \int 1^2 x + 1 (x - 2)^2 x - 2 \int 1^2 x + 1 (x - 2)^2 x - 2 \int 1^2 x + 1 (x - 2)^2 x + 1 (x - 2)^2 x - 2 \int 1^2 x + 1 (x 2 = 2 \ln|2x + 1| + x - 2 + 3 \ln|x - 2| + c 1 = \ln|2x + 1| + \dots + \ln|x - 2|3 + c x - 2 1 = \ln|(2x + 1)(x - 2)3| + \dots + c x - 2 2 c \text{ Let I} = \int \dots 2 dx 1 - x 2 2 A B dx$ = = + 1 - x2 (1 - x)(1 + x) 1 - x 1 + x So 2 ____ 1 Rewrite the intergral using the partial fractions. Note that using I saves copying the question again. Don't forget to divide by 2 when integrating 1 _____ and remember that the integral of 2x + 11 _____ does not involve ln. (x - 2)2 Simplify using the laws of logarithms. Remember that 1 - x2 can be factorised using the difference of two squares. 2 = A(1 + x) + B(1 - x) Let x = -1 then 2 = 2B so B = 1 Let x = 1 then 2 = 2A so A = 1 $9x2 - 4 \text{ Let } 9x2 - 3x + 2 \text{ I } = \int \\ 3x - 2 3x + 2 9x - 4 2 \text{ Let } x = - \\ 3 \text{ then } 8 = -4B \text{ so } B = -2 \text{ Factorise } 9x2 - 4 \text{ and then split into partial fractions. } 2 \text{ Let } x = - \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ then } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 3 \text{ the } 4 = 4A \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 4 \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 4 \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 4 \text{ so } A = 1 \text{ So } 2 1 \text{ I } = \\ 4 \text{ so } A = 1 \text{ so } 2 1 \text{ I } = \\ 4 \text{ so } A = 1 \text{ so } 2 1 \text{ so }$ numerator by the denominator. Example 21 $9x^2 - 3x + 2 dx$ Find \int $-3x I = \int (1 + dx 9x^2 - 4) 6 - 3x B A \equiv +$ 1 = x $= \int (1 + \frac{1}{2} + \frac{1}{2}) (1 + \frac{1}{2} + \frac{1}{2}) (2x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following
integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 1)(x - 2) 2 \text{ Find the following integrals. } 2(x + 3x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} (x + 3)(x - 1) x + 2 a \int \frac{1}{2} ($ 2, - < x < 2. (3 + 2x)(2 - x) 2 a Express f(x) in partial fractions. b Hence find the exact value of 17 - 5x dx, writing your answer in the form $\int 0$ $\int f(x) dx$, writing your answer as a single logarithm. c Find $\int f(x) dx$, giving your answer in the form ln k where k is a rational constant. 2 1 E/P 17 - 5x 3 4 f(x) = (3 + 2x)(2 - x) 29x 2 + 425 f(x) =, $x \neq \pm 39x 2 - 4$ (4 marks) (2 marks) (4 marks) (1 a + ln b, where a and b are constants to be found. E/P (3 marks) C B a Given that f(x) = A +, find the values of the constants A, B and C. (4 marks) 3x - 23x + 2b Hence find the exact value of Problem-solving $19x^2 + 43^2$ Simplify the

integral as much as possible before $\int -139x^2 - 4 \, dx$, writing your answer in the substituting your limits. form a + b ln c, where a, b and c are rational numbers to be found. (5 marks) 312 Integration E/P 6 + 3x - x 2, x>0 6 f(x) = x 3 + 2x 2 a Express f(x) in partial fractions. b Hence find the exact value of (4 marks) + 3x - dx, writing your answer in the form $a + \ln b$. (2) x 3 + 2x 2 46 x2 (5 marks) where a and b are rational numbers to be found. E/P C $32x 2 + 4 B 7 \equiv A + +$ 4x + 1 4x - 1 (4x + 1)(4x - 1) a Find the value of the constants A, B and C. (4 marks) 2 2 32x + 4 b Hence find the exact value of f dx writing your answer in the form 1 (4x + 1)(4x - 1) 2 + k ln m, giving the values of the rational constants k and m. (5 marks) 11.8 Finding areas You need to be able to use the integration techniques from this chapter to find areas under curves. Example 22 y 9 The diagram shows part of the curve y = $\sqrt{4}$ + 3x The region R is bounded by the curve, the x-axis and the lines x = 0 and x = 4, as shown in the diagram. Use integration to find the area of R. 9 2 1 1 x = x - 2 Remember Area $= \int x + 3x + 3 = 0$ and x = 4, as shown in the diagram. Use integration to find the area of R. 9 2 1 1 x = x - 2 Remember Area $= \int x + 3x + 3 = 0$ and x = 4, as shown in the diagram. Use integration to find the area of R. 9 2 1 1 x = x - 2 Remember Area $= \int x + 3x + 3 = 0$ and x = 4, as shown in the diagram. Use integration to find the area of R. 9 2 1 1 x = x - 2 Remember Area $= \int x + 3x + 3 = 0$ and x = 4, as shown in the diagram. Use integration to find the area of R. 9 2 1 1 x = 4 = 0 and x = 4. 9 _____ dx 0 \sqrt{4} + 3x y= __1 Substitute the limits. _ = 6(\sqrt{16} - \sqrt{4}) You don't need to give units when finding areas under graphs in pure maths. = 12 The area bounded by two curves can be found using integration: Area of R = \sqrt{a} (f(x) - g(x))dx = \sqrt{a} f(x)dx - \sqrt{a} g(x)dx b b y b f(x) g(x) R O a b Watch out You can only use this formula if the two curves do not intersect between a and b. x 313 Chapter 11 y Example 23 The diagram shows part of the curves y = f(x) and y = g(x), π where $f(x) = \sin x \cos 2x$, 0 < x < 2 The region R is bounded by the two curves. Use integration to find the area of R. Area = y = f(x) 1 y = g(x) R π 2 O fa (f(x) - g(x)) technology. Exercise 11H 1 Find the area of the finite region R bounded by the curve with equation y = f(x), the x-axis and the lines x = a and x = b. $\pi 2 c f(x) = \ln x$; a = 0, b = 131 + x $\pi e f(x) = x\sqrt{4 - x^2}$; a = 0, $b = 2 d f(x) = \sec x$ tan x; a = 0, b = -42 Find the exact area of the finite region bounded by the curve y = f(x), the x-axis and the lines x = a and x = b where: x 4x - 1 b f(x) = 2; a = 0, b = 2 a f(x) = 2; a = 0,x > 2 (x + 2)(2x - 1) Find the area of the shaded region bounded by the curve, the x-axis and the lines x = 1 and x = 2. (7 marks) $y \ge 1$ O 314 y = f(x) + x, x > 0. Find the area of the equation, y = f(x), 4x + 31 where f(x) =shaded region bounded by the curve, the x-axis and the lines x = 2 and x = 4. y = g(x) A O B C x a Write down the coordinates of points A, B and C. b Find the area of the shaded region. E/P (7 marks) Watch out Find the area of each region separately and then add the answers. Remember areas cannot be negative, so take the absolute value of any negative area. 6 The diagram shows a sketch of the curve, the x-axis and the line x = 2. a Use integration by parts to find $\int x^2 \ln x \, dx$. y $y = x^2 \ln x$ (3 marks) b Hence find the exact area of the shaded region, giving your 2 answer in the form ____ (a ln 2 + b), where a and b are integers. 3 (5 marks) O 2 x 315 Chapter 11 E/P 7 The diagram shows a sketch of the curve with equation $y = 3 \cos x \sqrt{\sin x} + 1$. $y D y = 3 \cos x \sin x + 1 R1 A O B C x R2$ a Find the coordinates of the points A, B, C and D. (3 marks) b Use a (5 marks) c Show that the regions R1 and R2 have the same area, and find the exact value of this $\sqrt{(3 \text{ marks})}$ area in the form a , where a is a positive integer to be found. P 8 f(x) = x2 and g(x) = 3x - x2 a On the same axes, sketch the graphs of y = f(x) and y = g(x), and find the suitable substitution to find $\int 3 \cos x \sqrt{\sin x} + 1 dx$. coordinates of any points of intersection of the two curves. b Find the area of the finite region bounded by the two curves. E/P 9 The diagram shows a sketch of part of the coordinates of the points A, B and C. (2 marks) bn b Find the area of region R1 in the form $a\sqrt{3} + c$, where a, b and c are integers to be found. (4 marks) c c Show that the ratio of R2 : R1 can be expressed as $(3\sqrt{3} + 2\pi) : (3\sqrt{3} - \pi)$. P (5 marks) 10 The diagrams show the curves $y = \sin \theta$, $0 < \theta < 2\pi$ and $y = \sin 2\theta$, $0 < \theta < 2\pi$. By choosing suitable limits, show that the total shaded area in the first diagram is equal to the total shaded area in the second diagram, and state the exact value of this shaded area. y 1 O -1 316 y y = sin θ 1 $2\pi \theta$ O -1 y = sin θ 1 $2\pi \theta$ O -1 y = sin x and y = cos x. y y = cos x A R1 1 y = sin x R3 R2 O $\pi \pi$ 2 x a Find the coordinates of point A. b Find the areas of: i R1 ii R2 iii R3 __ c Show that the ratio of areas R1 : R2 can be written as $\sqrt{2}$: 2. Challenge π The diagram shows the curves $y = \sin 2x$ and $y = \cos x$. $0 < x < _4$ y $y = \sin 2x$ and $y = \cos x$. $0 < x < _4$ y $y = \sin 2x$ and $y = \cos x$. $0 < x < _4$ y $y = \sin 2x$ and $y = \cos x$. $0 < x < _4$ y $y = \sin 2x$ and $y = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $y = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. $0 < x < _4$ y $x = \sin 2x$. $1 = \cos x$. 1method to approximate the area beneath a curve. Consider the curve y = f(x): y O y = f(x) a To approximate the area given by $\int a y \, dx$, you can divide b b x y the area up into n equal strips. Each strip will be of width h, b-a where h = _____n O a b h h h h x 317 Chapter 11 Next you calculate the value of y for each value of x that forms a boundary of one of the strips. So you find y for x = a, x = a + h, x = a + 2h, x = a + 3h and so on up to x = b. You can label these values y0, y1, y2, y3, ..., yn. Hint Notice that for n strips there will be n + 1 values of y. y y0 y1 y2 yn - 1 yn ... a O x b Finally you join adjacent points to form n trapezia and approximate the original area by the sum of the area of these n trapeziums. You may recall from GCSE maths that the area of a trapezium like this: y1 y0 h is given by _12 (y0 + y1) + b _1 h(y + y) + ... + _1 h(y + y) + ... + _1 h(y + y) + 2 Factorising gives: fa y dx \approx _12 h(y0 + y1 + y2 + y2 ... + yn - 1 $+y_n - 1 + y_n$) b fa y dx ≈ 12 h(y0 + 2(y1 + y2 ... + yn - 1) + y_n) b or This formula is given in the formula booklet but you will need to know how to use it. The trapezium rule: fab y dx ≈ 12 h(y0 + 2(y1 + y2 ... + yn - 1) + y_n) b - a where h = ____ n and y_i = f(a + ih) Example 24 The diagram shows a sketch of the curve y = sec x. The finite region R n is bounded by the curve, the x-axis, the y-axis and the line x = __ 3 The table shows the corresponding values of x and y for y = sec x n 3 x Integration n n a Complete the table with the values of y corresponding to x = __ and x = __, giving your answers 4 6 to 3 decimal places b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places. c Explain with a reason whether your estimate in part b will be an underestimate or an overestimate or an overestimate. $\pi 1 a \sec \alpha = \pi \approx 1.4144 \cos \alpha = 4 \times 0 \times 1616$ $12 6 1.035 1.155 \pi 4 1.414 \pi$ Substitute h = ____ and the five y-values into 12 the formula. $\pi 3 2 fa y dx \approx __21 h(y0 + 2(y1 + y2 ... + yn - 1) + yn)$ Online b Explore under- and overestimation when using the trapezium rule, using GeoGebra. $\pi 1 ____ I \approx __2(12)(1 + 2(1.035 + 1.155 + 1.414) + 2) \pi = ____ \times 10.208 24 = 1.336 224 075..... = 1.336 224 075.... = 1.336 224 075.... = 1.336 224 075.... = 1.336 224 075...$ 1.34 (2 d.p.) Problem-solving If f(x) is convex on the interval [a, b] then the trapezium rule will give an overestimate for c The answer would be above the curve, giving a greater answer than the real answer. fa f(x) dx. If it is concave then it b will give an ____ π 1 The diagram shows a sketch of the curve with equation y = $\sqrt{1 + \tan x}$, 0 < x < __ 3 y y= 1 + tan x I O π 3 x π π a Complete the table with the values for y corresponding to x = __ and x = __ 4 12 π π π π ___ _ _ x 0 4 3 6 12 y 1 1.2559 1.6529 (1 mark) 319 Chapter 11 π underestimate. ← Section 9.9 Exercise E 11I that I = $\int \sqrt{1 + \tan x \, dx}$, 0 b use the trapezium rule: $\pi \pi$ i with the values of y at x = 0, _ and _ to find an approximate value for I, giving your answer 3 6 to 4 significant figures; (3 marks) $\pi _ \pi \pi _ _$ ii with the values of y at x = 0, , and to find an approximate value for I, giving your 3 12 6 4 answer to 4 significant figures. (3 marks) E/P 2 The diagram shows the region R bounded by the x-axis and the $5\theta \pi \pi$ curve with equation $y = \cos _{,-} - - 64 < _{,-} 5525\theta$ The table shows corresponding values of θ and y for $y = \cos 5\theta 2 R - \pi 5\pi \theta 5 O a$ Complete the table giving the missing values for y to 4 decimal places. E (1 mark) b Using the trapezium rule, with all the values for y in the completed table, find an approximation for the area of R, giving your answer to 3 decimal places. (4 marks) e Calculate the percentage error in your answer in part b. (2 marks) 1 3 The diagram shows a sketch of the curve with equation y = 20.345 R b Use the trapezium rule, with all the values from your table, to estimate the area of the region R, giving your answer to 2 decimal places. (4 marks) E/P 4 The diagram shows the curve with equation $y = (x - 2) \ln x + 1$, x > 0. 1 1 1.5 0.7973 2 2.5 y $y = (x - 2) \ln x + 1$ 3 2.0986 R O 320 2 x O a Complete the table with the values of y corresponding to x = 2and x = 2.5. (1 mark) x y 1 ex + 1 1 3 x Integration Given that $I = \int ((x - 2) \ln x + 1) dx$, 3 1 b use the trapezium rule i with values of y at x = 1, 2 and 3 to find an approximate value for I, giving your answer to 4 significant figures. (3 marks) c Use the diagram to explain why an increase in the number of values improves the accuracy of the approximation. d Show by integration, that the exact value of $E/P \int 1 ((x - 2)\ln x + 1) dx$ is $3 3 - 2 \ln 3 (1 \max) (6 \max) + 4$. 5 The diagram shows the curve with equation $y = x\sqrt{2} - x$, 0 < x < 2. a Complete the table with the value of y corresponding to x = 1.5. (1 mark) x y 0 0 0.5 0.6124 1 1 1.5 y y=x 2-x 2 0 R Given that $I = \int x\sqrt{2} - x \, dx$, 2 0 O b use the trapezium rule with four strips to find an approximate value of leaving your (x - 3)(2x + 1) a Show that the coordinates of point A are (4, 0). 5 (4 marks) 0 1.6667 0.25 0.9697 0.5 0.6 0.75 1 0.1667 (2 marks) y = 4x - 5 (x - 3)(2x + 1) (1 mark) b Complete the table with the value answer in the form 2qp, where p and q are rational constants. 6 The diagram shows part of the curve with equation 4x - 5y =of y corresponding to x = 0.75. Give your answer to 4 decimal places. (1 mark) x y (5 marks) $\int 0 x \sqrt{2} - x dx$, 2 d Calculate the percentage error of the approximation in part b. E/P x 2 O 1.25 0 A 1 x 5 4 x - 5 4 Given that I = $\int 0^{10} x \sqrt{2} - x dx$ dx, 0 (x - 3)(2x + 1) c use the trapezium rule with values of y at x = 0, 0.25, 0.5, 0.75, 1 and 1.25 to find an dx, giving your answer in the form ln (). 0 (x - 3)(2x + 1) b (4 marks) e Calculate the percentage error of the approximation in part c. (2 marks) 321 Chapter 11 E/P 7 I = $\int e \sqrt{2x+1} dx 3 = 0$ approximate value for I, giving your answer to 4 significant figures. (3 marks) 5 4x - 5 a 4 d Find the exact value of a Given that v = e $\sqrt{2x+1}$, complete the table of values of y corresponding to x = 0.5, 1 and 1.5. x y 0 2.7183 0.5 1 1.5 2 9.3565 2.5 11.5824 (2 marks) 3 14.0940 b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the original integral, I, giving your answer to 4 significant figures. (3 marks) 2x + 1 to show that I may be expressed as $\int a$ kte t dt, giving the values of the constants a, b and k. b (5 marks) d Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures. (4 marks) E/P 8 a Given that y = cosec x, complete the table with values of y corresponding to $5\pi \pi 7\pi x = 0$, your answers to 5 decimal places. $12\ 2\ 12\ x\ y\ \pi$ 3 1.15470 5 π 12 π 2 7 π 12 (2 marks) 2 π 3 1.15470 b Use the trapezium rule, with all the values of y in the completed table, to obtain an 2 π estimate for $\int \pi \csc x \, dx$. Give your answer to 3 decimal places. $3\ 3\ 2\pi$ c Show that $\int \pi \csc x \, dx = \ln 3$. $3\ 3$ (4 marks) Hint For part c you may use the standard integral for cosec x from the formulae booklet: $\int cosec x \, dx = -\ln|cosec x + cot x| + c \, d$ Calculate the percentage error in using the estimate obtained in part b. (3 marks) (2 marks) 11.10 Solving differential equations. In this chapter you will solve first order differential equations by separating the variables. dy \blacksquare When $_ = f(x)g(y)$ you can write dx $_ \int 1 dy = \int f(x) dx g(y)$ Notation A first order differential equation contains a second order derivative, for d 2y example $_ 2 dx$ The solution to a differential equation will be a function. When you integrate to solve a differential equation you still need to include a constant of integration. It represents a family of solutions, all with differential equation you still need to include a constant of integrate to solve a differential equation. Integration dy For the first order differential equation = 12x 2 - 1, the general solution is y = 4x 3 - x + c, dx or y = x(2x - 1)(2x + 1) + c. y 2 c = 1 c = 0 c = -1 1 - 1 - 12 0 1 2 1 x - 1 Each of these curves represents a particular solution of the differential equation, for different values of the constant c. Together, the curves form a family of solutions y Online x Explore families of solutions using technology. -2 Example 25 dy Find a general solution to the differential equation $(1 + x^2) = x \tan y$. dx dy $x = 2 \tan y + x \sqrt{y} + x \sqrt{$ $\sin y = k\sqrt{1 + x^2}$ Finally remove the ln. Sometimes you might be asked to give your answer in the form y = f(x). This question did not specify that so it is acceptable to give the answer in the form y = f(x)g(y). dx Now separate the variables: 1 dy = f(x) dx g(y) 1 Use $\cot y = \int dx g(y) dx g(y) dx = \int dx g(y) dx g(y) dx g(y) dx$ $\tan y \int \cot x \, dx =$ ln|sin x| + c Don't forget the +c which can be written as ln k. Combining logs. Sometimes you are interested in one specific solution to a differential equation. You can find a particular solution to a differential equation. to the differential equation -3(v - 2) dv= dx (2x + 1)(x + 2) Hint The boundary condition in this question is that x = 1 when y = 4. given that x = 1 when y = 4. Leave your answer in the form y = f(x). 323 Chapter 11 -3 1 $\int_{----}^{----} dx y - 2 (2x + 1)(x + 2) -3 (2x + 1)(x + 2) - 3 (2x + 1)(x +$ $(2x + 1)(x + 2) A B \equiv$ + 1) (x + 2) - 3 = A(x + 2) + B(2x + 1) Let x = -2: -3 = -3B so B = 1 1 Let x = -2: 3 - 3 = -2 First separate the variables. Make sure the function on the left-hand side is in terms of y only, and the function on the left-hand side is in terms of y only. Convert the fraction on the RHS to partial fractions. So $2 1 1 \int$ Remove ln. Use the condition x = 1 when y = 4 by substituting these values into the general solution and solving to find k. Substitute k = 2 and write the answer in the form y = f(x) as requested. 11J 1 Find general solutions to the following differential equations. Give your answers in the form y = f(x). dy dy b ____ = y tan x a ____ = (1 + y)(1 - 2x) dx dx $y = y^2 \sin 2x d$ y = 2ex - y dx dx 2 Find particular solutions to the following differential equations using the given boundary conditions. dy dy $\pi \pi b$ y = 0, x = a $y = 2 \cos 2y \cos 2x$; y = 0, x = 0 d sin y cos x y = 0, x = 0 d sin y cos y x = 0 d sin y cos x y = 0, x = 0 d sin y cos y xgeneral solution to the differential equation dy x2 $_ = y + xy$, giving your answer in the form y = g(x). dx b Find the particular solution to the differential equation that satisfies the boundary condition y = e4 at x = -1. 324 Hint Begin by factorising the right-hand side of the equation. Integration E E/P E 4 Given that x = 0 when y = 0, find the particular solution to the differential equation dy (6 marks) $(2y + 2yx) = 1 - y^2$, giving your answer in the form y = g(x). dx dy 5 Find the general solution to the differential equation $e^x + y = 2x + xe^2$, giving your answer dx (6 marks) in the form $\ln |g(y)| = f(x)$. dy 6 Find the particular solution to the differential equation $(1 - x^2) = xy + y$, with boundary dx condition y = 6 at x = 0.5. Give your answer in the form y = f(x). (8 marks) E 7 E 8 E/P E E E dy Find the particular solution to the differential equation (1 + x 2) = x - xy 2, with boundary dx condition y = 2 at x = 0. Give your answer in the form y = f(x). (8 marks) E 7 E 8 E/P E E E dy Find the particular solution to the differential equation (1 + x 2)-y, with boundary condition dx y = ln 2 at x = 4. Give your answer in the form y = f(x). (8 marks) dy Find the particular solution to the differential equation ______ = cos 2y + cos 2x cos 2y, with dx $\pi \pi$ _____ boundary condition y = at x = . Give your answer in the form tan y = f(x). (8 marks) 4 dy π 10 Given that y = 1 at x = ____, solve the differential equation _______ = cos 2y + cos 2x cos 2y, with dx $\pi \pi$ ______ boundary condition y = at x = . Give your answer in the form tan y = f(x). equation = xy sin x. (6 marks) 2 dx 9 3x + 411 a Find f x dx, x > 0. $\sqrt{\sqrt{3xy + 4y}}$ Biven that y = 16 at x = 1, solve the differential equation giving = x dx your answer in the form y = f(x). dy 8x - 1812 a Express in partial fractions. (3x - 8)(x - 2) (2 marks) (6 marks) (3 marks) b Given that x > 3, find the general solution to the differential equation dy (5 marks) (x - 2)(3x - 8) = (8x - 18)y dx c Hence find the particular solution to this differential equation that satisfies y = 8 at x = 3, giving your answer in the form y = f(x). (4 marks) P E/P dy 13 a Find the general solution of ____ = 2x - 4. dx b On the same axes, sketch three different particular solutions to this differential equation. dy 1 14 a Find the general solution to the differential equation. c Write down the particular solutions to this differential equation. c Write down the particular solution that passes through the point (8, 3.1). (3 marks) (3 marks) (1 mark) 325 Chapter 11 E/P dy x 15 a Show that the general solution to the differential equation = - y can be written in dx 2 2 (3 marks) the form x + y = c. b On the same axes, sketch three differential equation. c Write down the particular solution that passes through the point (0, 7). (3 marks) (1 mark) 11.11 Modelling with differential equation. equations Differential equations can be used to model real-life situations. Example 27 The rate of increase of a population P of microorganisms at time t, in hours, is given by dP = 3P, k > 0 dt Initially the population was of size 8. a Find a model for P in the form P = Ae3t, stating the value of A. b Find, to the nearest hundred, the size of the population at time t = 2. c Find the time at which the population will be 1000 times its starting value. d State one limitation of this model for large values of t. a dP $_$ dt = 3P 1 dP = $\int 3 dt \int _ P \ln P = 3t + c P = e 3t$ separating the variables. Apply the laws of indices. ec is a constant so write it as A. You are told that the initial population was 8. This gives you the boundary condition P = 8 when t = 0. Substitute t = 2. $P = 1000 \times 8 = 8000 8000 = 8e$ 3t $1000 \approx 2.3$ hours = 2 h 18 mins d The population could not increase in size in this way forever due to limitations such as available food or space. 326 y Online Explore the solution to this example 28 Water in a manufacturing plant is held in a large cylindrical tank of diameter 20 m. Water flows out of the bottom of the tank through a tap at a rate proportional to the coube root of the volume. 3 dh a Show that t minutes after the tap is opened, $= -k\sqrt{h}$ for some constant k. dt b Show that the general solution to this differential equation may be written 3 as h = (P - Qt), where P and Q are constants. 2 h 20 m Initially the height of the water is 27 m. 10 minutes later, the height is 8 m. c Find the values of the constants. P and Q. d Find the time in minutes when the water is 27 m. 10 minutes later, the height of 1 m. a Use the formula for the values of the constants. P and Q. d Find the time in minutes when the water is 27 m. 10 minutes later, the height of 1 m. a Use the formula for the values of the constants. P and Q. d Find the time in minutes when the water is 20, so the radius is 10. V = $\pi r 2h = 100\pi h dV$ 3 = - c √ 100πh Problem-solving You need to use the information given in the question to construct a mathematical model. Water flows out at a rate proportional to the cube root of the volume. dV ______ is negative as the water is flowing out of the dt tank, so the volume is decreasing. dh ______ dh dV = ______ × $= 100 \pi \, dh \quad 3 \, dV \quad = - c \, \sqrt{V} \, dt$ $1 = x (-c \sqrt{100} \pi h) dt 100\pi = (100\pi) 3 - c \sqrt{100} \pi 3 \sqrt{h^2} h^2 3 = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = dt 100\pi b \int h^2 - c \sqrt{100} \pi dh So = -k \sqrt{h}, where k = -k \sqrt{h}, w$ = (9 - 10Q) 4 = 9 - 10Q 1 Q = 2 dh Use the chain rule to find dt dV dh Substitute for and dV dt dh $11 = dV = dV 100\pi dV$ dh c was the constant of proportionality and π is $3 c \times \sqrt{100\pi}$ constant so = k is a constant. 100 π Integrate this function by separating the variables. Let Q = 23 k and P = 23 c 2 Use 1 The rate of increase of a population P of rabbits at time t, in years, is given by = kP, k > 0. dt Initially the population was of size 200. a Solve the differential equations giving P in terms of k and t. (3 marks) b Given that k = 3, find the time taken for the population to reach 4000. (4 marks) c State a limitation of this model for large values of t. (1 mark) E/P 2 The mass M at time t of the leaves of a certain plant varies according to the differential equation dM _____ = M - M2 dt a Given that at time t = 0, M = 0.5, find an expression for M in terms of t. (5 marks) b Find a value of M when t = ln 2. (2 marks) c Explain what happens to the value of M as t increases. (1 mark) E/P 3 The thickness of ice x, in cm, on a pond is increasing at a rate that is inversely proportional to the square of the existing thickness of ice. Initially, the thickness of ice can be modelled by the equation x = $\sqrt{20t + 1}$. 3 7 b Find the time taken for the ice to increase in thickness from 2 cm to 3 cm. E/P (7 marks) (2 marks) 4 A mug of tea, with a temperature T °C is made and left to cool in a room with a temperature of 25 °C. The rate at which the tea cools is proportional to the difference in temperature of 25 °C. The rate at which the tea cools is proportional to the difference in temperature of 25 °C. The rate at which the tea cools is proportional to the difference in temperature of 25 °C. why k is a positive constant. (3 marks) Initially the tea is at a temperature of 85 °C. 10 minutes later the tea is at 55 °C. b Find the temperature, to 1 decimal place, of the surface area of a drop of oil, A mm2, at time t minutes can be modelled by the equation dA 3 = A 2 dt 10t2 Given that the surface area of the drop is 1 mm2 at t = 1, a find an expression for A in terms of t 400 b show that the surface area of the drop cannot exceed 361 328 (7 marks) mm2. (2 m time t, the depth of water in the bath tub is h cm. Water leaves the bottom of the bath through an open plughole at a rate of 500h cm3/min. dh a Show that t minutes after the tap has been opened, 60 ____ = 120 - 5h. (3 marks) dt When t = 0, h = 6 cm. b Find the value of t when h = 10 cm. E/P (5 marks) 1 7 a Express fractions. $P(10\ 000 - P)$ (3 marks) The deer population, P, in a reservation can be modelled by the differential equation dP 1 = $P(10\ 000 - P)$ dt 200 where t is the time in years since the study began. Given that the initial deer population is 2500, a (6 marks) b solve the differential equation dP 1 = $P(10\ 000 - P)$ dt 200 where t is the time in years since the study began. Given that the initial deer population is 2500, a (6 marks) b solve the differential equation giving your answer in the form P = -50t c Find the maximum deer population according to the model. (2 marks) E/P 8 Liquid is pouring into a container at a constant rate of 4 V cm 3 s -1, where V cm 3 is the volume of liquid in the container. dV a Show that - 4 = V - 160. (2 marks) dt Given that V = 5000 when t = 0, 1 b find the solution to the differential equation in the form V = a + be -4t, where a and b are constants to be found (7 marks) c write down the limiting value of V as $t \rightarrow \infty$. E/P (1 mark) 9 Fossils are aged using a process called carbon dating. The amount of carbon remaining in a fossil, R, decreases over time, t, measured in years. The rate of decrease expected carbon remaining. Determine the age of the fossil to the nearest year. (3 marks) Mixed exercise 11 1 By choosing a suitable method of integration, find: a d $\int (2x - 3) 7 dx \int x \ln x dx b e \int x \sqrt{4x} - 1 dx$ 4 sin x cos x dx \int 4 - 8 sin 2 x c f \int sin 2 x cos x dx 1 dx \int 3 - 4x 329 Chapter 11 2 By choosing a suitable method, evaluate the following definite integrals. Write your answers as exact values. a c e E/P 3 E/P 4 E/P 5 E 6 E 7 $\int -3x(x + 3) 5 dx = 0$ $\int 1 (16x + 32) b^2 dx = 0$ $\int 1 (16x + 32) b^2 dx = 0$ $2 16x + 8x - 34\pi$ for $x \sec 2x dx 4 d \int f \int 0\pi 3\pi 12 (\cos x + \sin x)(\cos x - \sin x) dx \ln 2$ 11 + ex dx e 12 a Show that $\int 2\ln x dx = 1 - e 1x (5 \text{ marks}) p 4p - 211 b$ $dx = 3 \ln 1 (x + 1)(2x - 1) p + 1 b 2 1 9$ Given $\int 1 (3 - 2) dx = 4$, find the value of b. x 2 Given $\int 0 \cos x \sin 3 x dx = 64$, where $\theta > 0$, find the smallest possible value of θ . $\theta 9$ Using the substitution t2 = x + 1, where x > -1, x dx. a find $\int \sqrt{x + 1.3 x} dx = 64$. Given that p > 1, show that \int _ dx. b Hence evaluate ∫0 $\sqrt{x} + 18$ (4 marks) (4 marks) (5 marks) (2 marks) (2 marks) a Use integration by parts to find fx sin 8x dx. (4 marks) b Use your answer to part a to find fx 2 cos 8x dx. E/P (5 marks) (4 marks) 5x2 - 8x + 1 f(x) = 2x(x - 1)2 C A B a Given that f(x) = + 2 find the values of the constants A, B and C. (4 marks) x x - 1 (x - 1) (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) x x - 1 (x - 1) (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of the constants A, B and C. (4 marks) fx 2 - 8x + 1 f(x) = 2 find the values of marks) b Hence find $\int f(x) dx$. c Hence show that $\int f(x) dx = \ln(3) - 249325$ (4 marks) 4 E/P 9 3 48 Given that y = x 2 + x, x > 0, dy a find the value of y which you found is a minimum. (3 marks) 48 The finite region R is bounded by the curve with equation y = x 2 + x, the lines x = 1, x = 4 and the x-axis. (2 marks) _3 c Find, by integration, the area of R giving your answer in the form $p + q \ln r$, where the numbers p, q and r are constants to be found. E/P 10 a Find $\int x 2 \ln 2x \, dx$. b Hence show that the exact value of 330 $\int 1 x 2 \ln 2x \, dx$ is $9 \ln 6 - 3215$ ____ 72 (4 marks) (6 marks) (4 marks) Integration E/P $3\pi 11$ The diagram shows the graph of $y = (1 + \sin 2x)^2$, 0 < x < 4 1 2 (4 marks) a Show that $(1 + \sin 2x) \equiv 2$ (3 + 4 sin $2x - \cos 4x$). b Hence find the area of the shaded region R. (4 marks) a Show that y = -3 at x = 0, solve the differential equation 4 dy x ex = dx sin 2y E 13 a Find $\int x \sin 2x \, dx$. dy π b Given that y = 0 at $x = _$, solve the differential equation $_$ = x sin 2x cos2y. 4 dx E/P $\exists \pi x 4$ (4 marks) (5 marks) the general solution to the differential equation dy $_$ dx = xy2, y > 0 (3 marks) b Given that y = 1 at x = 1, show 2y = 2, $-\sqrt{3} < x < \sqrt{33} - x$ is a particular solution to the differential equation. 2 The curve C has equation y = 2, $x \neq \pm \sqrt{33} - x$ c Write down an equation of the tangent to C at the point (1, 1), and find the coordinates of the point where it again meets the curve. (4 marks) E 15 a Using the substitution u = 1 + 2x, or otherwise, find $4x 1 \int dx$, $x \neq -2(1 + 2x)2 \pi b$ Given that y = when x = 0, solve the differential equation $4 dy x (1 + 2x)2 = dx \sin 2y (5 \text{ marks}) 331$ Chapter 11 E/P 16 The diagram shows the curve with equation $y = xe^{2x}$, $-2 < x < 211 y y = xe^{2x}$ The finite region R1 bounded by the curve, the x-axis and the 1 line x = -2 has area A1. The finite region R2 bounded by the curve, the x-axis and the 1 line x = -2 has area A2. E E/P a Find the exact values of A1 and A2 by integration. (6 marks) b Show that A1 : A2 = (e - 2) : e. (4 marks) O R1 1 2 (5 marks) dy b Use your answer to part a to find the solution to the differential equation ___ = x 2e 3 y - x, dx given that y = 0 when x = 0. Express your answer in the form y = f(x). (7 marks) 18 The diagram shows part of the curve y = e3x + 1 and the line y = 8. a Find h, giving your answer in terms of natural logarithms. 1 b Use integration to show the area of R is 2 + 3 ln 7. y 8 y = e3x + 1 y=8 (3 + 1) = 0. Content of the curve y = e3x + 1 and the line y = 8. a Find h, giving your answer in terms of natural logarithms. marks) The region R is bounded by the curve, the x-axis, the y-axis and the line x = h. 2 R (5 marks) h x O 19 a Given that C x 2 B = A + + x-1 x+1 x 2 - 1 find the values of the constants A, B and C. (4 marks) b Given that x = 2 at t = 1, solve the differential equation dx 2 = 2 - 2, x > 1 dt x You do not need to simplify your final answer. E/P x 17 a Find $\int x 2e - x \, dx$. The curve and the line intersect at the point (h, 8). E R2 - 12 20 The curve with equation $y = e^2 x - e^{-x}$, 0 < x < 1, is shown in the diagram. The finite region enclosed by the curve, the x-axis and the line x = 1 is shaded. (7 marks) $y = e^2 x - e^{-x}$, 0 < x < 1, is shown in the diagram. The finite region enclosed by the curve, the x-axis and the line x = 1 is shaded. values given to 5 decimal places as appropriate. x y 0 0 0.25 0.86992 0.5 2.11175 0.75 1 7.02118 a Complete the table with the missing value for y. Give your answer to 5 decimal places. R O 1 x (1 mark) b Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 4 decimal places. (3) marks) 332 Integration c State, with a reason, whether your answer to part b is an overestimate or an underestimate or an underestimate. (1 mark) d Use integration to find the exact value of R. Write your answer to part b. E/P (6 marks) (2 marks) 21 The rate, in cm3 s-1, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, V cm3, in the sump at that instant. At time t = 0, V = A. a By forming and integrating a differential equation, show that V = Ae-kt where k is a positive constant. b Sketch a graph to show the relation between V and t. 1 Given further that V = 2A at t = T, c show that $kT = \ln 2$. E/P (5 marks) (2 marks) dy x 22 a Show that the general solution to the differential equation = can be dx k - y written in the form x2 + (y - k)2 = c. (4 marks) b Describe the family of curves that satisfy this differential equation when k = 2. E/P y 23 The diagram shows a sketch of the curve y = f(x), 1 where $f(x) = 5 \times 2 \ln x - x + 2$, x > 0. The region R, shown in the diagram, is bounded by the curve, the x-axis and the lines with equations x = 1 and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and x = 4. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and $x \neq 1$ and $x \neq 1$. The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate. $x \neq 1$ and $x \neq 1$. The table below shows the corresponding values given to 4 decimal places as appropriate. $x \neq 1$ and $x \neq 1$. The table below shows the corresponding values given to 4 decim = f(x) R O 3.5 1.5693 1 4 x 4 2.4361 a Complete the table with the missing value of y. (1 mark) b Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 decimal places. (3 marks) c Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of R. (1 d Show that the exact area of R can be written in the form + ln e, where a, b, c, d b d and e are integers. (6 marks) e Find the percentage error in the answer in part b. (2 marks) E 24 a Find $\int x (1 + 2x^2) 5 \, dx$. π b Given that $y = _$ at x = 0, solve the differential equation 8 dy $_$ = $x(1 + 2x^2)5 \, cos^2 2y \, dx$ (3 marks) (5 marks) 333 Chapter mark) c a 11 E/P E/P 1 25 By using an appropriate trigonometric substitution, find $\int 2 \, dx \, 1 + x 26$ Obtain the solution to dy x(x + 2) = y, y > 0, x > 0 dx for which y = 2 at x = 2, giving your answer in the form $y^2 = f(x)$. (5 marks) (7 mark of the oil spill, in km2/day at time t days after it occurs is modelled as: dA t = k sin (), 0 < t < 12 3π dt dr t k (2 marks) a Show that = sin () 3π dt 2πr Given that the radius of the spill at time t = 0 is 1 km, and the radius of the spill at time t = π 2 is 2 km: b find an expression for r2 in terms of t (7 marks) c find the time, in days and hours to the nearest hour, after which the radius of the spill is 1.5 km. (3 marks) Hint Challenge Given $f(x) = x^2 - x - 2$, find: a 334 $\int -3 |f(x)| dx 3 D_r dx = x + c \int s dx = ex + c \int s dx = ex + c \int s dx = ex + c \int s dx = -\cos x + c \int s dx = ex + c \int s dx = ex + c \int s dx = -\cos x + c \int s dx = -\cos$ $c \int cosec^2 x \, dx = -\cot x + c n + 1 \int x^1 \, dx = \ln|x| + c \int sec^2 x = \tan x + c \int sec^2 x = \tan x \, dx = sec x + c \int f'(ax + b) \, dx = a^1 f(ax + b) + c^3 Trigonometric identities can be used to integrate expressions.$ dx, try $\ln|f(x)|$ and differentiate to check, and f(x) then adjust any constant. 5 To integrate an expression of the form $\int kf'(x)(f(x))n dx$, try (f(x))n dx, try (f(xand is called integration by substitution. dv du 7 The integration by parts formula is given by: $\int u dx = uv - \int v dx dx dx 8$ Partial fractions can be used to integrate algebraic fractions. 9 The area bounded by two curves can be found using integration: Area of $R = \int a (f(x) - g(x)) dx = \int a f(x) dx - \int a g(x) dx b b b 10$ The trapezium rule is: $\int a y dx \approx 10^{-10} dx = 1$ $12 h(y0 + 2(y1 + y2 ... + yn - 1) + yn) b b - a where h = n and yi = f(a + ih). dy 11 When = f(x)g(y) you can write dx 1 f dy = f(x) dx g(y) 335 12 Vectors Objectives After completing this chapter you should be able to: • Understand 3D Cartesian coordinates <math>\rightarrow$ pages 337-338 • Use vectors in three dimensions \rightarrow pages 339-343 • Use vectors to solve geometric problems \rightarrow pages 344-347 \bigcirc Model 3D motion in mechanics with vectors \rightarrow pages 347-349 Prior knowledge check 1 Given that p = 3i - j and q = -i + 2j, calculate: a 2p + q b $-3p + 4q \leftarrow$ Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, work out: a the magnitude of a b the unit vector that is parallel to a. \leftarrow Year 1, Section 11.2 2 Given that a = 5i - 3j, we have a = 5i - 3j. 11.3 3 M is the midpoint of the line segment AB. \rightarrow Given that AB = 4i + j, \rightarrow a find BM in terms of i and j. \leftarrow Year 1, Section 11.5 336 You can use vectors to describe relative positions in three dimensions. This allows you to solve geometrical problems in three dimensions and determine properties of 3D solids.

→ Mixed exercise Q9 Vectors 12.1 3D coordinates cartesian coordinate axes in three dimensions are written as (x, y, z). z Hint To visualise this, think of the x- and y-axes being drawn on a flat surface and the z-axis sticking up from the surface. y (0, 6, 0) 2 O -1 (3, 2, -1) 3 x You can use Pythagoras' theorem in 3D to find distances on a 3D coordinate grid. The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$. Example 1 Find the distance from the origin to the point P(4, -7, -1). √ 16 + 49 + 1 = √ 66 Substitute the values of x, y and z into the formula. You don't need to give units with distances on a coordinate grid. You can also use Pythagoras' theorem to find the distance between two points. The distance between the points (x1, y1, z1) and (x2, y2, z2) is $OP = \sqrt{42 + (-7)2 + (-1)2} =$ $AB = \sqrt{(1-8)2 + (3-6)2 + (4-(-5))2} = _$ $\sqrt{(x1 - x2)^2 + (y1 - y2)^2 + (z1 - z2)^2}$ Example 2 Find the distance between the points A(1, 3, 4) and B(8, 6, -5), giving your answer to 1 d.p. $\sqrt{(-7)} 2 + (-3) 2 + 9 2$ Be careful with the signs – use brackets when you substitute $AB = \sqrt{(5-4)} 2 + (0-2) 2 + (3-k) 2 = 3$ $= \sqrt{139} = 11.8$ (1 d.p.) 337 Chapter 12 Example 3 The coordinates of A and B are (5, 0, 3) and (4, 2, k) respectively. Given that the distance from A to B is 3 units, find the possible values of k. $\sqrt{1+4} + (9-6k+k2) = 31+4$ +9 - 6k + k2 = 9 k2 - 6k + 5 = 0 Problem-solving Use Pythagoras' theorem to form a quadratic equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equation in k. Square both sides of the equation is the equare both sis the equation is the equation is the eq the point P(2, 8, -4). 2 Find the distance from the origin to the point P(7, 7, 7). 3 Find the distance between A and B when they have the following coordinates: a A(3, 0, 5) and B(1, -1, 3) P 4 The coordinates of A and B are (7, -1, 2) and (k, 0, 4) respectively. Given that the distance from A to B is 3 units, find the possible values of k. P 5 The coordinates of A and B are (5, 3, -8) and (1, k, -3) respectively. Given that the distance from A to B is 3 10 units, find the possible values of k. Challenge a The points A(1, 3, -2), B(1, 3, 4) and C(7, -3, 4) are three vertices of a solid cube. Write down the coordinates of A and B are (5, 3, -8) and (1, k, -3) respectively. the remaining five vertices. An ant walks from A to C along the surface of the cube. b Determine the length of the shortest possible route to describe position and displacement relative to the x-, y- and z-axes. You can represent 3D vectors as column vectors or using the unit Find the position vectors of A and B in ijk notation. \rightarrow b Find the vector AB as a column vector. $\rightarrow \rightarrow a$ OA = i + 5j - 2k, OB = -3j + 7k $\rightarrow \rightarrow \rightarrow b$ AB = OB - OA 0 - 1 1 = -3 - 5 = -8 (7) (-2) (9) Example The position vector of a point is the vector from the origin to that point. -3j + 7k. When adding and subtracting vectors is parallel to 4i - 5k. 42ai 4a + b = 4-3 and b = -2. (0) (5) a Find: i 4a + b = 4-3 and b = -2. (0) (5) a Find: i 4a + b = 4-3 $+ -2(5)(0) 48 = -12 + -2(20)(0) 12 = -14(20) Use the rules for scalar multiplication and addition of vectors: <math>\lambda p p u p + u \lambda q = \lambda q and q + v = q + v(r)(\lambda r)(r)(w)(r + w) 339$ Chapter 12 4 2 ii 2a - 3b = 2 - 3 - 3 - 2(5)(0) 4 12 = -6 - -6(10)(0) - 8 = 0(10) | | /4 12 - 14 _ b i 4a + b = -14 = 3 3(20) _ 20 \sqrt{3} 4 0(-5) 4a + b is not parallel to 4i - 5k 4 - 8 ii 2a - 3b = 0 = -2 0 (10) (-5) which is not a multiple of 4 0 (-5) 2a - 3b is parallel to 4i - 5k. Watch out 4i - 5k = 4i + 0j - 5k. Make sure you include a 0 in the j-component of the column vector. 6 Find the magnitude of a = 2i - j + 4k and hence find \hat{a} , the unit vector in the direction of a. The magnitude of a is given by $|a| = \sqrt{22} + (-1)2 + 42$ Use Pythagoras' theorem. = $\sqrt{21} a 1$ $(2i - j + 4k) a = \hat{} = |a| \sqrt{21142} i - j +$ write this as $\sqrt{21}\sqrt{21}\sqrt{21}\sqrt{21}$ Online Check your answer using the vector functions on your calculator. z y a You can find the angle between a given vector and any of the coordinate axes by considering the appropriate right-angled triangle. O $\theta x x = xi + yj + zk$ makes an angle θx with the positive x x-axis then $\cos \theta x =$ and similarly for the angles θy and $\theta z |a|$ 340 Hint This rule also works with vectors in two dimensions. Vectors Example 7 Find the angles that the vector a = 2i - 3j - k makes with each of the positive coordinate axes to 1 d.p. ______ $|a| = \sqrt{22 + (-3)3 + (-1)2} = \sqrt{4 + 9 + 1} = \sqrt{142 x} = 0.5345... \cos \theta x = ___ = 0.5345...$ First find |a| since you will be using it three times. $\theta y = 143.3^{\circ}(1 \text{ d.p.}) - 1z = -0.2672... \cos \theta z = = |a| \sqrt{14}$ The formula also works with negative components. The y-component is negative, so the vector makes an obtuse angle with the positive y-axis. $\theta x = 57.7^{\circ}(1 \text{ d.p.}) y - 3 = -0.8017... \cos \theta z = = |a| \sqrt{14}$ The formula also works with negative components. The y-component is negative, so the vector makes an obtuse angle with the positive y-axis. $\theta x = 57.7^{\circ}(1 \text{ d.p.}) y - 3 = -0.8017... \cos \theta z = = |a| \sqrt{14}$ The formula also works with negative components. d.p.) Example Write down at least 4 d.p., or use the answer button on your calculator to enter the exact value. 8 The points A and B have position vectors 4i + 2j + 7k and 3i + 4j - k relative to a fixed origin, $O. \rightarrow Find AB$ and show that nOAB is isosceles. $4 \rightarrow 3 \rightarrow OA = a = 2$, $OB = b = 4(-1)(7) \rightarrow AB = b - a = Write down the position vectors of the position vectors of the position vectors of the position vectors of the position vectors <math>4i + 2j + 7k$ and 3i + 4j - k relative to a fixed origin, $O. \rightarrow Find AB$ and show that nOAB is isosceles. $4 \rightarrow 3 \rightarrow OA = a = 2$, $OB = b = 4(-1)(7) \rightarrow AB = b - a = Write down the position vectors of the p$ A and B. $4 - 1 \ 3 \ 4 - 2 = 2 \ (-1) \ (7) \ (-8) \rightarrow |AB| = \sqrt{(-1) 2 + 2 2 + (-8) 2} \rightarrow Use \ AB = b - a. = \sqrt{69} \ (-1)$ A and B. $4 - 1 \ 3 \ 4 - 2 = 2 \ (-1) \ (7) \ (-8) \longrightarrow |AB| = \sqrt{(-1) \ 2 + 2 \ 2 + (-8) \ 2}$ parallel to 6i + 4j + 10k. 2 The vectors a and b are defined by a = 341 Chapter 12 P p 1 3 The vectors a and b are defined by a = 2 and b = q . (r) (-4) Given that a + 2b = 5i + 4j, find the magnitude of: a 3i + 5j + k b 4i - 2k d 5i - 9j - 8k e i + 5j - 7k c i + j - k 5 7 2 5 Given that p = 0, q = 1 and r = -4, find in column vector form: (2) (2) (-3) a p+q b q-r d 3p - r e p - 2q + r c p+q+r \rightarrow 6 The position vector of the point A is 2i - 7j + 3k and AB = 5i + 4j - k. Find the possible values of t. P 8 Given that a = 5i + 2tj + tk, and that |a| = $3\sqrt{10}$, find the possible values of t. 9 The points A, B and C have coordinates (2, 1, 4), (3, -2, 4) and (-1, 2, 2). a Find, in terms of i, j and k: i the point (-1, 3, -5). Find: \rightarrow a the vector PQ b the distance between P and Q \rightarrow c the unit vector in the direction of PQ $\rightarrow \rightarrow 11$ OA is the vector 4i - j - 2k and OB is the vector -2i + 3j + k. Find: $\rightarrow a$ the vector AB b the distance between A and B $\rightarrow c$ the unit vector in the direction of AB. 342 Vectors 12 Find the unit vector in the direction of AB . 342 Vectors 12 Find the unit vector in the direction of AB . 342 Vectors 12 Find the unit vector in the direction of each of the following vectors. The points A, B and C have position vectors -7, -3 and -6 respectively. (4) (3) (3) $\rightarrow \rightarrow \rightarrow a$ Find the vectors AB, AC and BC. $\rightarrow \rightarrow \rightarrow b$ Find |AB |, |AC | and |BC | giving your answers in exact form. c Describe triangle ABC. E (3 marks) (6 marks) (1 mark) 14 A is the point (3, 4, 8), B is the point (1, -2, 5) and C is the point (7, -5, 7). $\rightarrow \rightarrow \rightarrow a$ Find the vectors AB, AC and BC. (3 marks) b Hence find the lengths of the sides of triangle ABC. (6 marks) c Given that angle ABC = 90° find the size of angle BAC. (2 marks) 15 For each of the given vectors, 3 4 (7) find the angle made by the vector with: P E/P 2 0 (-3) a -i + 7j + k b c i the positive x-axis ii the positive y-axis iii the positive z-axis 16 A scalene triangle has the coordinates (2, 0, 0), (5, 0, 0) and (4, 2, 3). Work out the area of the triangle PQR. \rightarrow Given that PQ = 3i - j + 2k and $\rightarrow QR = -2i + 4j + 3k$, show that $\angle PQR = 78.5^{\circ}$ to 1 d.p. R P Q (5 marks) Challenge Find the acute angle that the vector a = -2i + 6j - 3k makes with the x-y plane. Given your answer to 1 d.p. 343 Chapter 12 12.3 Solving geometric problems You need to be able to solve geometric problems involving vectors in three dimensions. Example 9 A, B, C and D are the points (2, -5, -8), (1, -7, -3), (0, 15, -10) and (2, 19, -20) respectively. $\rightarrow \rightarrow a$ Find AB and DC, giving your answers in the form pi + qj + rk. $\rightarrow \rightarrow b$ Show that the lines AB and DC are parallel and that DC = 2AB. c Hence describe the quadrilateral ABCD. $\rightarrow \rightarrow \rightarrow a$ AB = OB - OA = $(i - 7j - 3k) - (2i - 5j - 8k) = -i - 2j + 5k \rightarrow \rightarrow DC = OC - OD = (15j - 10k) - (2i + 19j - 20k) = -2i - 4j + 10k \rightarrow b$ 2AB = $2(-i - 2j + 5k) \rightarrow = -2i - 4j + 10k = DC$ So AB is parallel to DC and half as long. c There are two unequal parallel sides, so ABCD is a trapezium. Example 10 Watch out AB refers to the line segment \rightarrow between A and B (or its length), whereas AB refers to the vector from A to B. Note that $\rightarrow \rightarrow AB = BA$ but $AB \neq BA$. Problem-solving If you can't work out what shape it is, draw a sketch showing AB and DC. C D A B Online Explore the solution to this example visually in 3D using GeoGebra. P, Q and R are the points (4, -9, -3), (7, -7, -7) and (8, -2, -0) respectively. Find the coordinates of the point S so that PQRS forms a parallelogram. S R(8, -2, -0) respectively. Find the coordinates of the point S so that PQRS forms a parallelogram. S R(8, -2, -0) respectively. the positions on the sketch don't correspond to the real positions in 3D - it is still a helpful way to visualise the problem. $Q(7, -7, -7) \rightarrow \rightarrow So OS = OR + RS = OR + QP 8 \rightarrow OR = -2$ and (0) 7 4 - 3 $\rightarrow \rightarrow \rightarrow QP = OP - OQ = -9 - -7 = -2$ (-3) (-7) (4) 5 - 3 8 $\rightarrow So OS = -2 + -2 = -4$ (0) (4) (4) which means that S is the point (5, -4, 4) 344 You could also go from O to S via P: $\rightarrow \rightarrow \rightarrow OS = OP + PS \rightarrow \rightarrow = OP + QR$ Vectors In two dimensions you saw that if a and b are two non-parallel vectors and pa + qb = ra + sb then p = r and q = s. In other words, in two dimensions with two vectors you can compare coefficients on both sides of an equation. In three dimensions you have to extend this rule: Notation If a, b and c are vectors in three dimensions which do not all lie on the same plane. Non-coplanar vectors are vectors which are not in the same plane. In particular, since the vectors i, j and k are non-coplanar, if pi + qj + rk = ui + vj + wk then p = u, q = v and r = w. Example 11 Given that 3i + (p + 2)j + 120k = pi - qj + 4pqrk, find the values of p, q and r. Comparing coefficients of j gives p = 3. Comparing coefficients of j gives p + 2 = -q so q = -(3 + 2) = -5. Comparing coefficients of k gives $120 \ 120 = 4$ pqr so r = = -2 4 × 3 × (-5) Since i, j and k do not lie in the same plane you can compare coefficients. When comparing coefficients like this just write the coefficients. For example, write 3 = p, not 3i = pi. Example 12 The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. Vectors a, b and c are the position vectors of the vertices A, B and C respectively. Prove that the diagonals OE and BG bisect each other. F Hint Bisect means 'cut into two equal parts'. In this case you need to prove that both diagonals are bisected. E Problem-solving C B G c D b O a A Suppose there is a point of intersection, H, of OE and BG. $\rightarrow \rightarrow OH = rOE$ for some scalar r. $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ But OH = OB + BH and BH = sBG $\rightarrow \rightarrow \rightarrow$ for some scalar s, so OH = OB + sBG. If there is a point of intersection, H, it must lie on both diagonals. You can reach H directly from O (travelling along OE), or by first travelling to B then travelling along BG. Use this to write two $\rightarrow \rightarrow$ expressions for OH. $\rightarrow \rightarrow$ H lies on the + c - b So (1) becomes r(a + b + c) = b + s(a + c - b) Comparing coefficients in a and b gives r = s and r = 1 - s 1 Solving simultaneously gives r = s = 2 These solutions also satisfy the coefficients of c so the lines do intersect at H. 1 OH = 2 OE so H bisects OE. 1 BH = 2 BG so H bisects BG, as required. Exercise P a, b and c are three noncoplanar vectors so you can compare coefficients. In order for the lines to intersect, the values of r and s must satisfy equation (1) completely: 1(a + b + c) = b + 1(a + c - b) 2 2 The coefficients of a, b and c all match so both ways of writing the vector OH are identical. Vector proofs such as this one often avoid any coordinate geometry, which tends to be messy and complicated, especially in three dimensions. 12C 10 1 4 1 The points A, B and C have position vectors -4, 4 and 0 relative to a fixed origin, O. (8) (7) (30) a Show that: $\rightarrow \rightarrow \rightarrow \rightarrow i$ |OA | = |OB | ii |AC | = |BC | b Hence describe the quadrilateral OACB. P 2 The points A, B and C have coordinates (2, 1, 5), (4, 4, 3) and (2, 7, 5) and (3, respectively. a Show that triangle ABC is isosceles. b Find the area of triangle ABC. c Find a point D such that ABCD is a parallelogram. P 3 The points A, B, C and D have coordinates (7, 12, -1), (11, 2, -9), (14, -14, 3) and (8, 1, 15) respectively. a Show that ABCD is a parallelogram. P 3 The points A, B, C and D have coordinates (7, 12, -1), (11, 2, -9), (14, -14, 3) and (8, 1, 15) respectively. guadrilateral ABCD. P 4 Given that (3a + b)i + j + ack = 7i - bj + 4k, find the values of a, b and c. P 5 The points A and B have position vectors 10i - 23j + 10k and pi + 14j - 22k respectively, relative to a fixed origin O, where p is a constant. Given that (\OAB is isosceles, find three possible positions of point B. E/P 6 The diagram shows a triangle three-dimensional figure formed by six parallelograms. The diagram shows a parallelopiped with vertices O, A, B, C, D, E, F, and G. $\rightarrow \rightarrow \rightarrow a$, b and c are the vectors OA, B, C, D, E, F, and G. $\rightarrow \rightarrow \rightarrow a$, b and c are the vectors of the vertices A, B and C respectively. The point M lies on OE such that OM : ME = 3 : 1. The straight line AP passes through point M. Given that AM : MP = 3 : 1, prove that P lies on the line EF and find the ratio FP : PE : PF C B E M c G D b O Challenge a A 2 -5 1 1 a, b and c are the vectors 0, 0 and 3 respectively. Find scalars (4) (-3) (1) 28 p, q and r such that pa + qb + rc = -12 (-4) 2 The diagram shows a cuboid with vertices O, A, B, C, D, E, F and G. M is the midpoint of FE and N is the midpoint of AG. a, b and c are the position vectors of the vertices A, B and C respectively. Prove that the lines OM and BN trisect the diagonal AF. M F E C B G c D N b O a Hint Trisect means divide into three equal parts. A 12.4 Application to mechanics 3D vectors can be used to model problems in mechanics in the same way as you have previously used 2D vectors. Example 13 A particle of mass 0.5 kg is acted on by three forces: F1 = (2i - j + 2k) N F2 = (-i + 3j - 2k) N F3 = (4i - 3j - 2k) N F3 = (4It is easier to add vectors in column vector form. Use the vector form of F = ma. This is Newton's second law of motion. \leftarrow Statistics and Mechanics Year 1, Chapter 10 c |a| = $\sqrt{102 + (-2)2 + (-6)2} = \sqrt{140}$ m s-2. - 6K) m s-2 at 2 $1 \vee 140 \times 62 = 2$ 35 m Exercise Because the particle starts from rest the acceleration acts in the same direction as the motion of the particle. So it is moving in a straight line with constant acceleration. You can use the suvat formulae to find the distance travelled. \leftarrow Statistics and Mechanics Year 1, Chapter 9 12D 1 A particle is acted upon by forces of (3i - 2j + k) N, (7i + 4j + 3k) N and (-5i - 3j) N. a Work out the resultant force R. b Find the exact magnitude of the resultant force. P 2 A particle, initially at rest, is acted upon by a force that causes the particle. 3 A body of mass 4 kg is moving with a constant velocity when it is acted upon by a force of (2i - 5j + 3k) N. a Find the acceleration of the body while the force acts. b Find the magnitude of this acceleration to 3 s.f. P 4 A particle is acceleration at (2i - k) m s-2, find F2, giving your answer in the form (pi + qj + rk) N. P 5 A particle of mass 2 kg is in static equilibrium and is acted upon by three forces: F1 = (i - j - 2k) N F2 = (-i + 3j + bk) N F3 = (aj - 2k) N F2 = (-i + 3j + bk) N F3 = (aj - 2k) N F2 = (-i + 3j + bk) N F3 = (aj - 2k) N F2 = (-i + 3j + bk) N F3 = (aj - 2k) N F2 = (-i + 3j + bk) N F3 = (aj - 2k) N F3 = s-2 d the magnitude of this acceleration e the angle the acceleration vector makes with the unit vector j. 348 Vectors 6 In this question i and j are the unit vector set and north, and k is the unit vector set and north, and k is the unit vector set and north and air resistance F, acting on the aeroplane are modelled as: $T = 2800i - 1800j + 300k L = 11\ 000k F = -900i + 500j$ a Taking $g = 9.8\ m s - 2$, find the magnitude of the aeroplane. b Determine whether the aeroplane is ascending, and find the size of the obtuse angle its acceleration makes with the vector k. Mixed exercise 12 P 1 The points A(2, 7, 3) and B(4, 3, 5) are joined to form the line segment AB. The point M is the midpoint of AB. Find the distance from M to the point C(5, 8, 7). P 2 The coordinates of P and Q are (2, 3, a) and (a - 2, 6, 7). Given that the distance from P to Q is $\sqrt{14}$, find the possible values of a. P P $\rightarrow \rightarrow \rightarrow 3$ AB is the vector -3i + tj + 5k, where t > 0. Given that $|AB| = 5\sqrt{2}$, show that AB is parallel to 5 6i - 8j - 2 tk. 4 P is the point (2, -2, 1) and R is th $(4) \rightarrow \rightarrow \rightarrow a$ Find the vectors DE, EF and FD. $\rightarrow \rightarrow \rightarrow b$ Find |DE|, |EF| and |FD| giving your answers in exact form. c Describe triangle DEF. E E/P (3 marks) (6 marks) (1 mark) 6 P is the point (-6, 2, 1), Q is the point (1, 3, -2). $\rightarrow \rightarrow \rightarrow a$ Find the vectors PQ, PR and QR. (3 marks) b Hence find the lengths of the sides of triangle PQR. (6 marks) c Given that angle QRP = 90° find the size of angle PQR. (2 marks) C 7 The diagram shows the triangle ABC. $\rightarrow \rightarrow$ Given that AB = -i + j and BC = i - 3j + k, find $\angle ABC$ to 1 d.p. A B (5 marks) 349 Chapter 12 E/P 8 The diagram shows the quadrilateral ABCD. 15 6 $\rightarrow \rightarrow$ Given that AB = -2 and AC = 8, find the area of the (11) (5) quadrilateral. (7 marks) P C B D A 9 A is the point (2, 3, -2), B is the point (0, -2, 1) and C is the point D. b Give the mathematical name for the shape ABCD. c Work out the area of ABCD. P C 10 The diagram shows a tetrahedron OABC. a, b and c are the position vectors of A, B and C respectively. B P, Q, R, S, T and U are the midpoints of OC, AB, OA, BC, OB and AC respectively. P U c Prove that the line segments PQ, RS and TU meet at a point and bisect each other. b O P S a T Q R A 11 A particle of mass 2 kg is acted upon by three forces: F1 = (bi + 2j + k) N F2 = (3i - bj + 2k) N F3 = (-2i + 2j + (4 - b)k) N Given that the particle accelerates at 3.5 m s-2, work out the possible values of b. P 12 In this question i and j are the unit vectors due east and due north respectively, and k is the unit vector acting vertically upwards. A BASE jumper descending with a parachute is modelled as a particle of mass 50 kg subject to forces describing the wind, W, and air resistance, F, where: W = (20i + 16j) NF = (-4i - 3j + 450k) Na With reference to the model, suggest a reason why the k components. b Taking g = 9.8 m s - 2, find the resultant force acting on the BASE jumper. c Given that the BASE jumper starts from rest and travels a distance of 180 m before landing, find the total time of the descent. 350 Vectors Challenge A student writes the following hypothesis: If a, b and c are three non-parallel vectors in three dimensions, then pa + qb + rc = sa + tb + uc \Rightarrow p = s, q = t and r = u Show, by means of a counter-example, that this hypothesis is not true. 1 The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2} + z^2 2$ The distance between the points (x1, y1, z1) and (x2, y2, z2) is $\sqrt{(x1 - x^2)^2 + (y1 - y^2)^2 + (z1 - z^2)^2}$ 3 The unit vectors along the x-, y- and z-axes are denoted by i, j and k respectively. 1 0 0 j = 1 k = 0 (0) (0) (1) p Any 3D vector can be Summary of key points written in column form as pi + qj + rk = q(r) 45x If the vector a = xi + yj + zk makes an angle θx with the positive x-axis then $\cos \theta x =$ ____ and |a| similarly for the angles θy and θz . If a, b and c are vectors in three dimensions which do not all lie in the same plane then you can compare their coefficients on both sides of an equation. 351 3 E/p Review exercise 1 E/p 1 A curve has equation of the tangent at E/p x = 2 is $y - (3e 6 - 1)x - 2 + \ln 4 + 5e 6$ = 0 (6) e $3x \leftarrow$ Section 9.2 E/p 3 A curve has equation 3 2 y = - 2, $x \neq (4 - 6x)$ 3 Find an equation of the normal to the curve at x = 1 in the form ax + by + c = 0, where a, b and c are integers. (7) E 4 A curve C has equation y = (2x - 3) 2e 2x. dy a Use the product rule to find (3) dx b Hence find the coordinates of the stationary points of sin x dy a Use the quotient rule to find (3) dx b Show that the equation of the π tangent to the curve at x = is 2 2 π y = (π - 2)x + (1 -) (4) 4 \leftarrow Section 9.5 352 7 Assuming standard results for sin x and cos x, prove that the derivative of arcsin x 1 is C. (3) \leftarrow Section 9.4 E 1) 2 (x - 5 The curve C has equation y = $(5)\sqrt{1-x}$ $2 \leftarrow$ Section 9.6 \leftarrow Section 9.3 E 6 a Show that if y = cosec x then dy = -cosec x cot x (4) dx dy b Given x = cosec 6y, find in terms dx of x. (6) 8 A curve has parametric equations $\pi x = 2 \text{ cot } t$, $y = 2 \sin 2 t$, 0, t < 2 dy a Find in terms of t. (3) dx b Find an equation of the tangent to the π curve at the point where t = (3) 4 c Find aCartesian equation of the curve in the form y = f(x). State the domain on which the curve is defined. (3) \leftarrow Section 9.7 E/p 9 The curve C has parametric equations 1 1 x = ____, y = ____, -1 < t < 1 1+t 1-t The line l is a tangent to C at the point 1 where t = _2 a Find an equation for the line l. (5) b Show that a Cartesian equation for the x curve C is $(3) 2x - 1 \leftarrow$ Section 9.7 Review exercise 3 E/p 10 A curve C is described by the equation E a Show that there is a root α of p(x) = 0 in the interval [1.7, 1.8]. (2) $3x^2 - 2y^2 + 2x - 3y + 5 = 0$ Find an equation of the normal to C at the point (0, 1), giving your answer in the form ax + by + c = 0, where a, b and c are integers. (7) b By considering a change of sign of f(x) in a suitable interval, verify that $\alpha = 1.746$ correct to 3 decimal places. (3) \leftarrow Section 10.1 11 A set of curves is given by the equation sin $x + \cos y = 0.5$ E a Use implicit differentiation to find an dy expression for (4) dx For $-n < x < \pi$ and $-\pi < y < \pi$ b find the coordinates of the points dy where = 0. dx x = ln (3x - 5) + 2, x > 3 5 The iterative formula xn+1 = ln (3xn - 5) + 2, x = 4 is used to find a value for α . (3) b Calculate the values of x 1, x = nd x = 10 (3x - 5) + 2, x = 4 is used to find a value for α . (3) b Calculate the values of x = 10 (3x - 5) + 2, x = 3 5 The iterative formula xn+1 = ln (3x - 5) + 2, x = 10 cm3, where $V = 3 \pi r3$. dV a Find _____(1) dr The volume of the balloon increases with time t seconds according to the formula dV ______(3) dt \leftarrow Section 9.10 E (2) The root of f(x) = 0 is α . y = x 2e - x, x < 0 E/p 16 f(x) = e x - 2 - 3x + 5 a Show that the equation f(x) = 0can be written as \leftarrow Section 9.8 E/p 15 p(x) = cos x + e - x 14 g(x) = x 3 - x 2 - 1 a Show that there is a root α of g(x) = 0 in the interval [1.4, 1.5]. (2) b By considering a change of sign of g(x) in a suitable interval, verify that $\alpha = 1.466$ correct to 3 decimal places. (3) E 1 17 f(x) = ____3 + 4x 2, $x \neq 2$ (x - 2) a Show that there is a root α of f(x) = 0 in the interval [0.2, 0.3]. (2) b Show that the equation f(x) = 0 can be _______ -1 written in the form x = +2. (3) $4x2 \sqrt{c}$ Use the iterative formula _______ -13 ______ +2, x0 = 1 to calculate the $xn+1 = 4xn 2 \sqrt{v}$ alues of x1, x2, x3 and x4 giving your answers to 4 decimal places. (3) d By considering the change of sign of f(x) in a suitable interval, verify that $\alpha = 1.524$ correct to 3 decimal places. (2) \leftarrow Section 10.1 353 Review exercise 3 E/p 18 The diagram shows part of the curve with equation $\overline{y} = f(x)$, where 1 f(x) = x 2e x - 2x - 10. The point A, 10 with x-coordinate a, is a stationary point on the curve. The equation f(x) = 0 has a root α in the interval [2.9, 3.0]. y x O d Taking x 0 = 0.3 as a first approximation to α , apply the Newton-Raphson process once to find f(x) to obtain a second approximation to α . Give your answer to 3 decimal places. (4) \leftarrow Sections 10.2, 10.3 E/p 20 The value of a currency x hours into a 14-hour trading window can be modelled by the function 2x 2x v(x) = 0.12 cos () - 0.35 sin () + 120 5 5 where 0 < x < 14. y B A A a Explain why x0 = a is not suitable to use as a first approximation to α . (1) b Taking x = 0 as a first approximation to α , apply the Newton-Raphson process to find an approximation to α . (2) \leftarrow Section 10.3 E/p 3 1 2 19 f(x) = $x 3 - x 3 + -4, x \neq 0 x 10$ a Show that there is a root α of f(x) = 0 in the intervals i [0.2, 0.3] (1) ii [2.6, 2.7] (1) b Show that the equation f(x) = 0 can be written in the form 3 10 1 2 4 + x 3 - (3) x= x) 3(\sqrt{c} Use the iterative formula, 3 101 24 + xn 3 - , x0 = 2.5 to xn+1 =turning point in the interval [4.7, 4.8]. (1) d Taking x = 12.6 as a first approximation, apply the Newton- Raphson method once to y(x) to obtain a second approximation for the time when the share index is a maximum. Give your answer to 3 decimal places. (3) e By considering the change of sign of y(x) in a suitable interval, verify that the xcoordinate at point B is 12.6067, correct to 4 decimal places. (2) \leftarrow Sections 7.5, 9.3, 10.4 Review exercise 3 E/p 21 Given $\int (12 - 3x)2 \, dx = 78$, find the value E/p a of a. (4) 3 28 The diagram shows a sketch of part of the curve with equation $y = 8 \sin x \cos 3 x$. $y \leftarrow$ Section 11.2 E/p 22 a By expanding $\cos (5x + 2x)$ and $\cos (5x - 2x)$ using the double angle formulae, or otherwise, show that $\cos 7x + \cos 3x \equiv 2 \cos 5x \cos 2x$. (4) b Hence find $\int 6 \cos 5x \cos 2x \, dx \, y = 8 \sin x \cos 3x \, R$ (3) \leftarrow Sections 7.1, 11.3 E/p m 3 23 Given that $\int mx \, 3e x \, 4 \, dx = (e \, 81 - 1), 4 \, 0$ find the value of m. (3) 24 Using the substitution u = 2x - 1, or otherwise, find the exact value of 5 3x dx (6) [1 $\sqrt{2x-1} \leftarrow$ Sections 11.4, 11.8 2 E 29 The following is a table of values for $\sqrt{y} = 1 + \sin x$, where x is in radians. x y \leftarrow Section 11.5 E 1 2 x 3 1 (1 – x2) 2 (6) dx 0 1 0.5 1.216 1 p 1.5 1.413 2 q a Find the missing values for p and q in the table, to 4 decimal places. (2) b Using the trapezium rule, with all the values for y in the completed table, find an approximation for I, where 25 Use the substitution u = 1 - x2 to find the exact value of $\int 0 x$ Find the area of the shaded region R. (4) \leftarrow Section 11.4 E π 2 O \leftarrow Section 11.5 I = $\int \sqrt{1 + \sin x} \, dx$, giving your 2 $E/p \ 26 \ f(x) = (x \ 2 \ 1 \ 1) \ln x$ Find the exact value of $0 \ \int 1 \ e \ f(x) \ dx$. (7) \leftarrow Section 11.9 \leftarrow Section 11.6 $E/p \ 27 \ a \ Express \ 5x + 3$ dx, giving your (2x - 3)(x + 2) answer as a single logarithm. R (4) \leftarrow Section 11.7 O 0.2 0.4 0.6 0.8 1 x The diagram shows the graph of the curve with equation y = xe2x, x > 0. The finite region R (2x - 3)(x + 2) fractions. (4) answer to 3 decimal places. E/p 30 y in partial (3) b Hence find the exact value of 6 5x + 3 $\int 2$ bounded by the lines x = 1, the x-axis and the curve is shown shaded in the diagram. 355 Review exercise 3 E/p a Use integration to find the exact area of R. (4) The table shows values of x and y between 0 and 1. x 0 0.2 0.4 0.6 0.8 1 2x 1.99207 7.38906 y = xe 0 0.29836 b Find the missing values in the table. (1) c Using the trapezium rule, with all the values for y in the completed table, find an approximation for the area of R, giving your answer to 4 significant figures. (4) d Calculate the percentage error in your answer in part c. (2) - Sections, 11.6, 11.9 E/p 2x - 1 31 a Express in partial (x - 1)(2x - 3) fractions. (4) b Given that x > 2, find the general solution of the differential equation dy (4) (2x - 3)(x - 1) = (2x - 1)y dx c Hence find the particular solution of this differential equation that satisfies E/p y = 10 at x = 2, giving your answer in the form y = f(x). (2) - Sections 11.7, 11.10 E/p 32 A spherical balloon is being inflated in such a way that the rate of increase of its volume, V cm3, with respect to time dV k t seconds is given by $_{--}$ = $_{--}$, where k is a dt V positive constant. Given that the radius of the balloon is 4 r cm, and that V = $_{-3}$ m³, a prove that r satisfies the differential equation of the differential equation of the balloon is 4 r cm, and that V = $_{-3}$ m³, a prove that r satisfies the differential equation of the balloon is 4 r cm, and that V = $_{-3}$ m³, a prove that r satisfies the differential equation of the balloon is 4 r cm. pouring into a container at a constant rate of 20 cm3 s-1 and is leaking out at a rate proportional to the volume of the liquid already in the container. a Explain why, at time t seconds, the volume, V cm3, of liquid in the container satisfies the differential equation dV ____ = 20 - kV dt where k is a positive constant. (2) The container is initially empty. b By solving the differential equation, show that V = A + Be - kt giving the values of A and B in terms of k. (5) dV Given also that = 10 when t = 5, dt c find the volume of liquid in the container at 10 s after the start. (3) \leftarrow Sections 11.10, 11.11 34 The rate of decrease of the concentration of a drug in the blood stream is proportional to the concentration C of that drug which is present at that time. The time t is measured in hours from the administration of the drug and C is measured in micrograms per litre. a Show that this process is described by dC the differential equation = -kC, dt explaining why k is a positive constant. (2) b Find the general solution of the differential equation in the form C = f(t). (4) After 4 hours, the concentration of the drug in the bloodstream is reduced to 10% of its starting value C0. c Find the exact value of k. (3) - Sections 11.10, 11.11 Review exercise 3 E/p 35 The coordinates of P and Q are (-1, 4, 6) and (8, -4, k) respectively. Given that the distance from P to Q is 7√ 5 units, find the possible values of k. (3) E 40 A particle of mass 2 kg is in equilibrium and is acted upon by 3 forces: F 1 = (ai + 2j - 4k) N F 2 = (-9i + 5j + ck) N \leftarrow Section 12.1 E/p F 3 = (3i + bj + 5k) N a Find the values of a, b and c. 36 The diagram shows the triangle ABC. (2) F1 is removed. Work out: B A C \rightarrow Given that AB = -i + 6j + 4k and $\rightarrow AC = 5i - 2j - 3k$, find the size of \angle BAC to one decimal place. (5) b the resultant force R acting on the particle. (1) c the acceleration of the particle, giving your answer in the form (pi + qj + rk) m s-2 (2) d the magnitude of the acceleration. (2) \leftarrow Section 12.2 E 37 P is the point (-6, 3, 2) and Q is the point (4, -2, 0). Find: \rightarrow (1) a the vector PQ b the unit vector in the direction of \rightarrow PQ (2) \rightarrow c the angle PQ makes with the positive z-axis. (2) \rightarrow The vector AB = 30i - 15j + 6k. d Explain, with a reason, whether the $\rightarrow \rightarrow$ vectors AB and PQ are parallel. (2) \leftarrow Section 12.2 E/p 38 The vertices of triangle MNP have coordinates M(-2, 0, 5), N(8, -5, 1) and P(k, -2, -6). Given that triangle MNP is isosceles and k is a positive integer, find the value of k. (4) Challenge 1 The curve C has implicit equation ay + x + 4xy = y + 2 a Find, in terms of a where = 0. dx b Show that there does not exist a point dy \leftarrow Section 9.8 where = 0. dx 2 The diagram shows the curves $y = \sin x + 2$ and $3\pi y = 0$.

 $\cos 2x + 2$, 0 < x < 2 Find the exact value of the total shaded area on the diagram. $y = \sin x + 2y = \cos 2x + 2 \leftarrow$ Section 12.3 O $x \leftarrow$ must have: Mathematical Formulae and Statistical Tables, Calculator 1 2 3 4 π A curve C has parametric equations x = sin 2 t, y = 2 tan t, 0 < t, 2 dy Find in terms of t. dx Find the set of values of x for which a 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c both 2(7x - 5) - 6x < 10x - 7 (2) b |2x + 5| - 3 > 0 (4) c |2x - 5| - 3 > 0 (4) c |2x - 5| - 3 > 0 (4) c |2x - 5| - 3 > 0 (4) c |2x - 5| - 3 > 0 (4) c |2x - 5| - 3 > 0 (4) c |2x - 5| y - 3 = 0 does not intersect the circle with equation $x^2 + kx + y^2 + 4y = 4$ a Show that 5x 2 + (k - 20)x + 17 > 0. (4) b Find the range of possible values of k. Write your answer in exact form. (3) Prove, for an angle θ measured in radians, that the derivative of $\cos \theta$ is $-\sin \theta$. You may assume the compound angle formula for $\cos (A \pm B)$, and that $\sin h$ $x^2 + 4x - 2$ normal to the curve at R intersects the curve again at a point T. Find the coordinates of T, giving your answers in their simplest form. (6) 7 A geometric series is 96 and the sum to infinity of the series is 600. (4) a Show that $25r^2 - 25r + 4 = 0$. b Find the two possible values of r. (2) For the larger value of r: c find the corresponding value of a. (1) (3) d find the smallest value of n for which S n exceeds 599.9. 8 The diagram shows the graph of f(x). The points B and D are stationary points of the graph of f(x). The points B and D are stationary points of the graph of f(x). The points B and D are stationary points of the graph of f(x). The points B and D are stationary points of the graph of f(x). The points B and D are stationary points of the graph of f(x). -f(x) + 5 c y = 2f(x - 3) 9 (3) (3) (3) Find all the solutions, in the interval $0 < x < 2\pi$, to the equation 31 - 25 cos x = 19 - 12 sin 2 x, giving each solution to 2 decimal places. (5) 10 The value, V, of a car decreases over time, t, measured in years. The rate of decreases in value of the car at that time. (4) a Given that the initial value of the car is V 0, show that V = V 0e -kt The value of the car after 2 years is £25 000 and after 5 years is £15 000. (3) b Find the exact value of k and the val and D. The distances, in miles, between each pair of cities, as measured in a straight-line, are labelled on the diagram. A new road is to be built between cities B and D. A 8 B 11 D 21 19 14 C a What is the minimum possible length of this road? Give your answer to 1 decimal place. b Explain why your answer to part a is a minimum. (7) (1) 12 A footballer takes a free-kick. The path of the ball towards the goal can be modelled by the equation y = -0.01x 2 + 0.22x + 1.58, x > 0, where x is the horizontal distance from the goal in metres and y is the height of the ball in metres. The goal is 2.44 m high. (3) a Rewrite y in the form A - B(x + C) 2, where A, B and C are constants to be found. b Using your answer to part a, state the distance from goal at which the ball is at the greatest height and its height at this point. (2) c How far from goal is the football when it is kicked? (2) d The football is headed towards the goal. The keeper can save any ball that crosses the goal is the football is headed towards the football is headed towards the goal. kick will result in a goal. (2) 13 A box in the shape of a rectangular prism has a lid that overlaps the box is h cm. The height of the box is h cm. The box and lid can be created exactly from a piece of cardboard of area 5356 cm2. The box has volume, V cm3. 3 cm Height Width Length a Show that V = _3 (2678x - 9x 2 - 2x 3) Given that x can vary b use differentiation to find the positive value of X, to 2 decimal places, for which V is stationary. c Prove that this maximum value, e determine the percentage of the area of cardboard that is used in the lid. 2 360 (5) (4) (2) (1) (2) Exam-style practice Mathematics A Level Paper 2: Pure Mathematics Time: 2 hours You must have: Mathematical Formulae and Statistical Tables, Calculator 1 The graph of $y = ax^2 + bx + c$ has a maximum at (-2, 8) and passes through (-4, 4). Find the values of a, b and c. (3) 2 The points P(6, 4) and Q(0, 28) lie on the straight line 11 as shown. y Q 12 P R O x 11 a Work out an equation for the straight line 12. (2) c Work out the coordinates of R. (2) d Work out the area of \triangle PQR. (3) 3 The function f is defined by f : x \rightarrow e3x - 1, Find f-1(x) and state its domain. 4 A student is asked to solve the equation $\log 4(x + 3) + \log 4(x + 4) = 2$ The student's attempt is shown. $x \in \mathbb{R}$. (4) 1 1 log $4(x + 3) + \log 4(x + 4) = 2(x + 3) + (x + 4) = 2x + 7 = 2x = x = 22 - 55 - 2361361$ Exam-style practice a Identify the error made by the student. b Solve the equation correctly. (1) (4) 5 The function p has domain -14 < x < 10, and is linear from (-14, 18) to (-6, -6) and from (-6, -6) to (10, 2). a Sketch y = p(x). (2) b Write down the range of p(x). (1) c Find the values of a, such that p(a) = -3. (2) 6 f(x) = x3 - kx2 - 10x + k a Given that (x + 2) is a factor of f(x), find the value of k. (2) b Hence, or otherwise, find all the solutions to the equation f(x) = x3 - kx2 - 10x + k a Given that (x + 2) is a factor of f(x), find the value of k. (2) b Hence, or otherwise, find all the solutions to the equation f(x) = x3 - kx2 - 10x + k a Given that (x + 2) is a factor of f(x), find the value of k. (2) b Hence, or otherwise, find all the solutions to the equation f(x) = x3 - kx2 - 10x + k a Given that (x + 2) is a factor of f(x). 0, leaving your answers in (4) the form $p \pm \sqrt{q}$ when necessary. 7 In $\triangle DEF$, DE = x - 3 cm, DF = x - 10 cm and $\angle EDF = 30^\circ$. Given that the area of the triangle is 11 cm², (3) a show that x satisfies the equation $x^2 - 13x - 14 = 0$ b calculate the value of x. (2) 8 9 3 π The curve C has parametric equations $x = 6 \sin t + 5$, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, $y = 6 \cos t - 2$, - < t < 10 + 5, - = 10 + 5, - = 10 +3 4 2 2 a Show that the Cartesian equation of C can be written as (x + h) + (y + k) = c, where h, k and c are integers to be determined. (4) b Find the length of C. Write your answer in the form pn, where p is a rational number to be found. (3) 4x + 7x $2(x-2)(x+4) C B \equiv A + + x-2x+4$ a Find the values of the constants in ascending powers of x, as far as the term in x2. (x - 2)(x + 4) Give each coefficient as a simplified fraction. (6) $\rightarrow \rightarrow 10$ OAB is a triangle. OA = a and OB = b. The points M and N are midpoints of OB and BA respectively. The triangle midsegment theorem states that 'In a triangle A, B and C. (4) 2 4x + 7x b Hence, or otherwise, expand the line joining the
midpoints of any two sides will be parallel to the third side and half its length.' B M N O Use vectors to prove the triangle midsegment theorem. 362 A (4) Exam-style-practice 11 The diagram shows the region R bounded by the x-axis and the curve with equation $3\pi y = x^2(\sin x + \cos x), 0 < x < 4y = x^2(\sin x + \cos x), 3\pi 4 O x$ The table shows corresponding values of x and y for $y = x^2(\sin x + \cos x)$. x 0 π _ 3 π _ 2.08648 0 2 8 4 a Copy and complete the table giving the missing values for y to 5 decimal places. (1) b Using the trapezium rule, with all the values for y in the completed table, find an approximation for y to 5 decimal places. the area of R, giving your answer to 3 decimal places. (4) c Use integration to find the exact area of R, giving your answer to 3 decimal places. (6) d Calculate, to one decimal places. (1) 12 Ruth wants to save money for her newborn daughter to pay for university costs. In the first year she saves £1000. Each year she plans to save £150 more, so that she will save £1150 in the second year, £1300 in the third year, and so on. a Find the amount that Ruth will have saved over the 18 years. (3) Ruth decides instead to increase the amount she saves by 10% each year. c Calculate the total amount Ruth will have saved after 18 years under this scheme. π 13 a Express 0.09 cos x + 0.4 sin x in the form R cos (x - α), where R > 0 and 0 < α < 2 Give the value of α to 4 decimal places. (4) (4) The height of a swing above the ground can be modelled using the equation 16.4 h = , 0 < t < 5.4, where h is the height of the swing, in cm, and t_t 0.09 cos () + 0.4 sin (_) 2 2 t is the time, in seconds, since the swing was initially at its greatest height. 363 Exam-style practice b Calculate the minimum value of t, to 2 decimal places, when this minimum value of t, to 2 decimal places the swing was initially at its greatest height. when the swing is at a height of exactly 100 cm. (4) 14 The diagram shows the height, h, in metres of a rollercoaster during the first few seconds of the ride. The graph is y = h(t), where h(t) = -10e-0.3(t-6.4) + 70. $y \land y = h(t)$ O t a Find h9(t). 3e-0.3(t-6.4) + 5b Show that when h9(t) = 0, t = ln (find an approximation for the t-coordinate of A, the iterative formula 3e-0.3(tn - 6.4) 5 tn + 1 = ln () + 6.4 is used. 4 8 364 c Let t0 = 5. Find the values of t1, t2, t3 and t4. Give your answers to 4 decimal places. (3) d By choosing a suitable interval, show that the t-coordinate of A is 5.508, correct to 3 decimal places. laces. (2) Answers Answers CHAPTER 1 Prior knowledge 1 1 2 3 a (x - 1)(x - 5) x - 3 a x + 6 a even b (x + 4)(x - 4) c (3x - 5)(3x + 5) x + 4 x + 5 b c - 3x + 1 x + 3 b either c either d odd Exercise 1A 1 2 3 4 5 B At least one multiple of three is odd. a At least one multiple of three is odd. a At least one prime number between 10 million and 11 million. c If p and q are prime numbers there exists a number of the form (pq + 1) that is not prime. d There is a number of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the above statements are true. a There exists a number of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is either not prime or not a multiple of 3. e None of the form 2n - 1 that is ei contradicts the assumption that n2 is odd. Therefore if n2 is odd then n must be odd. a Assumption: there is a greatest even integer 2n. 2(n + 1) > 2n 2n + 2 = even + even = even So there exists an even integer 2n. 2(n + 1) > 2n 2n + 2 = even + even = eventhere exists a number n such that n3 is even but n is odd, n is odd so write n = 2k + 1 n3 = $(2k + 1)3 = 8k3 + 12k2 + 6k + 1 = 2(4k3 + 6k2 + 3k) + 1 \Rightarrow n3$ is odd, p = 2k + 1 q is odd, q = 2m + 1 $1 pq = (2k + 1)(2m + 1) = 2km + 2k + 2m + 1 = 2(km + k + m) + 1 \Rightarrow pq$ is odd. This contradicts the assumption that pq is even. Therefore, if pq is even, p = 2k q is even, p = 2k q is even, p = 2k + 2m = 2(k + m) \Rightarrow so p + q is even. Therefore, if pq i assumption that p + q is odd. Therefore, if p + q is odd that at least one of p and q is odd. a Assumption: if ab is an irrational, $b = _$ where c and d are integers. f ce ab = _____, ce is an integer, df is an integer. df 6 7 Therefore ab is a rational number. This contradicts assumption that ab is irrational. Therefore if ab is an irrational number that at least one of a and b is an irrational number then neither a nor b is irrational. c a is rational, a = where c and d are integers. d e b is rational, b = where c and f are integers. f cf + de df cf, de and df are integers. So a + b is rational. This contradicts the assumption that a + b is irrational. Therefore if a + b is irrational. So now 3a + 2b = 17 3a is also an integer. 2b is also an integer. The sum of two integers will always be an integer, so 3a + 2b = 17. Therefore there exists no integers a and b for which 21a + 14b = 1. a Assumption: There exists a number n such that n_2 is a multiple of 3, but n is not a multiples of 3. We know that all multiples of 3 can be written in the form n = 3k + 1 and 3k + 2 are not multiples of 3. Let n = 3k + 1 $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 4k + 1) + 1$ In this case n^2 is not a multiple of 3. Let $n = 3k + 2m^2 = (3k + 2)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 4k + 1) + 1$ In this case n^2 is not a multiple of 3. Let $n = 3k + 2m^2 = (3k + 2)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 4k + 1) + 1$ In this case n^2 is not a multiple of 3. This contradicts the assumption that n2 is a multiple of 3. Therefore if n2 is a multiple of 3, n is a multiple of 3. Therefore if n2 is a multiple of 3, n is a multiple of 3, be a multiple of 3. We know from part a that this means a must also be a multiple of 3. Write a = 3c, which means a2 = (3c)2 = 9c2. Now 9c2 = 3b2, or 3c2 = b2. Therefore b2 must be a multiple of 3, which implies b is also a multiple of 3. If a and b are both multiples of 3, this contradicts the statement that there are no common factors between a and b. Therefore, $\sqrt{3}$ is an irrational number. 365365 Answers 8 Assumption: there is an integer solution to the equation $x^2 - y^2 = 2$. Remember that $x^$ Exercise 1C This contradicts the statement that there is an integer solution to the equation $x^2 - y^2 = 2$. Therefore the original statement must be true: There are no integer solutions to the equation $x^2 - y^2 = 2$. Between b a and b. $3a^2 = 2a^2 + 3a^2 + 3a^2$ 3 or a3 = 2b3 b This means that a3 is even, so a must also be even. If a is even, a = 2n. So a3 = 2b3 becomes (2n)3 = 2b3 which means 8n3 = 2b3 or 4n3 = b3 or 2(2n3) = b3. This means that b3 must be even, so b is also even. If a and b are both even, they will have a common factor of 2. This contradicts the statement that a and b have no common factors. 3 We can conclude the original statement is true: $\sqrt{2}$ is an irrational number. 1 10 a m could be non-positive, e.g. if n = 2 b Assumption: There is a least positive rational number, n. a n = -2 b Assumption: There is a least positive rational number. 1 10 a m could be non-positive, e.g. if n = -2 b Assumption: There is a least positive rational number. 1 a m could be non-positive, e.g. if n = -2 b Assumption: There is a least positive rational number. 1 a m could be non-positive, e.g. if n = -2 b Assumption: There is a least positive rational number. 1 a m could be non-positive rational number. 1 a m could be non-positive rational number. the least positive rational number. Therefore, there is no least positive rational number. Exercise 1B 1 2 3 4 5 6 a 2 1 a _ b a c _ 2 cd a - 3 1 a _ b _ x - 2 2(a + 3) x 1 f 4 g _ e _ x + 5 6 r 5 1 4 f _ d _ e _ 2 2 10 x y + 1 x - 3 _ c d _ y y 2(x + y) 2 3y - 2 _ i h 2 (x - y) 2 x - 8 x - 8 All factors cancel exactly except $a \div 2 6x - \overline{8 6x} - \overline{8 3x} + 14x - 24 \overline{2 (2x + 1)(x - 2)} (3x - 4)(x + 6) + 14x - 24 3x \times$ 2x + 10 1 - 4e 2x2 - 3x - 2 2x2 - 3x - 2 = = -1 8 - x - (x - 8) a = 5, b = 12 x - 4 20e + 4 ab x =x-2 x-2 2(3x - 4) (2x + 1)(x + 6)x-2 = (x - 1)(x + 2) 8x - 2c= 2 2 2 9 13; f9(4) = __ b f9(x) = 2x + __ 2 2 366366 3-x e ___ x2 2a - 15 f ____ 10b 7 a __ 12 2 x+3 a ___ 22 366366 3-x e ___ x2 2a - 15 f ____ 10b 7 a ___ 12 2 x+3 a ___ 10b 7 a ___ 12 2 x+3 a ___ 10b 7 a ___ 12 2 x+3 a ___ 10b 7 a ___ 12 2 x+3 a ___
10b 7 a ___ 12 2 x+3 a ___ 10b 7 a ___ 10 a x(x + 1) - x + 7 b6 2x – 4 e 2(x + 4) f x - y = 1 and $x + y = 2 \Rightarrow x = 32$, $y = 12 \cdot x - y = -2$ and $x + y = -1 \Rightarrow x = -2$ (2x + 1)(x - 1) - x - 5 d(x - 2)(x + 2) - x - 7 c6(x + 3)(x - 1) x + 3 a2(x + 1) 3x + 1 b2(x + 1)(x + 3) 3x + 3y + 2d(y - x)(y + x) 2x + 5 e(x + 2) 2(x + 1) 7x + 8 f2(x + 3)(x - 4) 4 2x - 19x(x + 1)(x + 2) (x + 5)(x - 3) - x 2 - 24x - 8b3x(x-2)(2x+1) 9x 2 - 14x - 7c(x - 1)(x + 1)(x - 3) = 650x + 37366ax+ + 5 6x 2 + 14x + 6 a x + 2 x 2 - 2x - 8 (6x + 1)(6x - 1) x(x + 2)(x - 4) 6(x - 4) 36 =+ x+3x-332q - xx+42A = 12, B = 32345A = 24, B = -2A = 1,(x + 2)(x - 4) (x + 2)(x - 4) (x + 2)(x - 4) x 3 - 2x 2 - 2x + 12 =(x + 2)(x - 4) b Divide x3 - 2x2 - 2x + 12 by (x + 2) to give x2 - 4x + 6 Exercise 1D 1 4 2 a ____ + ____ x+3 x-2 3 5 c ___ - ____ 2x x - 4 2 4 e x 2x + 1 3x - 2 3 6 2 cB = -2, C = 3 D = -1, E = 2, F = -5 6 7 3 1 b ____ - ___ x+1 x+4 4 1 d 2x + 1x - 321 - - x + 1x - 431h - x + 5x - 3f312a - + x + 1x - 2x + 5521b - + 2x + 5520b - + 2x + 550b - + 2x + 550b - + 2x + 500b - + 500b - + 2x + 500b - + 500b x+1 x+2 x-5 5 3 2 a - + x x+1 x-1 -1 2 b $\overline{2x+5}(x+5) = 246D = 3$, E = -2, F = -4C = 3, D = 1, E = 2A = 2, B = 4, C = 116D5x + 4 2x - 1 Challenge 6 1 2 x-2x+1x-3 Full worked solutions are available in SolutionBank. Online Answers Exercise 1E 1 3 5 7 8 A = 0, B = 1, C = 3 P = -2, Q = 4, R = 2 A = 2, B = -4 A = 4, B = 1 and C = 12. 19 4 a - $2 \times 2x - 1$ (2x - 1) This contradicts assumption that q2 is irrational. Therefore if q2 is irrational then q is irrational then q is irrational. $a_134a513$, $c = -5aa = _{34}$, $b = -_{2}$ 8 Exercise 1F 1 2 3 4 5 6 7 8 9 10 A = 1, B = 1, C = 2, D = -6a = 2, b = -3, c = 5, d = -10 p = 1, q = 2, r = 4 m = 2, n = 4, p = 7 A = 4, B = 1, C = -8 and D = 3. A = -3. A 4, B = -13, C = 33 and D = -27 p = 1, q = 0, r = 2, s = 0 and t = -6 a = 2, b = 1, c = 1, d = 5 and e = -4 A = 3, B = -4, C = 1, D = 4, E = 1 a (x2 - 1) (x2 + 1) = (x - 1)(x + 1)(x2 + 1), a = 1, b = -1, c = 1, d = 0 and e = 1. Exercise 1G 1 2 3 4 5 6 7 A = 1, B = -2, C = 3 A = 1, 5, D = 1A = 1, B = 5, C = -5A = 2, B = -4, C = 133221a4 + bx + - 2(x - 1)(x + 4)x(x - 2)(x - 2)83473, D = A = 2, B = -3, C = 11119A = 2, B = -1, C = 5, D = -, E = 83.3 Mixed exercise 1 12 Assume $\sqrt{12}$ is a rational number. a Then $\sqrt{12}$ for some integers a and b. b Further assume that this fraction is in its simplest terms: there are no common factors between a and b. a2 So 0.5 = ___2 or 2a2 = b2. b Therefore b2 must be a multiple of 2. Write b = 2c, which means e6 - 22x - 4 x - 41337bg9(x) = 32x - , g9(-2) = - 88676x + 18x + 52 x2 - 3x - 10 3 12 x + integers a2 and b2 are integers. So q2 is rational. $2(x^2 + 4)(x - 5)$ b $(x^2 - 7)(x + 4) \ 3 \ 2x + 3 \ c \ x \ 4(e^6 - 1) \ b$ x - 1 x 2 + 2x - 3 x(x + 3)(x - 1) 3(x + 3) 12 =(x + 3)(x - 1)x + 38101213141516A = 3, B = -29P = 1, Q = 2, R = -3D = 5, E = 211A = 4, B = -2, C = 3D = 2, E = 1, F = -2A = 1, B = -4, C = 3, D = 8A = 2, B = -4, C = 6, D = -11A = 1, B = 0, C = 1, C = 1,(x + 3)(x - 1) (x + 3)(x - 1) (x + 3)(x - 1) (x 2 + 3x + 3)(x - 1) x 2 + 3x + 3 =D = 3A = 1, B = 2, C = 3, D = 4, E = 1. 17 A = 2, B = -94, C = -14 18 P = 1, Q = -12, R = -52 19 a f(-3) = 0 or f(x) = (x + 3)(2x + 1)(x + 1) Challenge Assume L is not perpendicular to OA. Draw the line through O which is perpendicular to L. This line meets L at a point B, outside the -(x + 3)(2x + 1)(x + 1) Challenge Assume L is not perpendicular to OA. Draw the line through O which is perpendicular to L. This line meets L at a point B, outside the circle. Triangle OBA is right-angled at B, so OA is the hypotenuse of this triangle, so OA > OB. This gives a contradiction, as B is outside the circle, so OA < OB. Therefore L is perpendicular to OA. CHAPTER 2 Prior knowledge $2 2 5p - 8x 9 - 5x y = 7 2 a 25x^2 - 30x + 5 or 5(5x^2 - 6x + 1) 3x + 7 1 b$ 8 + 9x a y b y = ex y y = x(x + 4)(x - 5) 1 O x - 4 O 5 x 367367 Answers $c y 9 y = sinx 1 O 90^{\circ} 180^{\circ} 270^{\circ} 360^{\circ} a y O - 5 a 28 b 0 1 a 34 2 3 4 a 5 b 46 c 40 a 16 b 65 c 0 a Positive |x| graph with vertex at (1, 0), y-intercept at (0, 1) b Positive |x| graph with vertex at (-1, 12, 0), y-intercept at (0, 3) 19 d$ Positive |x| graph with vertex at (0, 7) d Positive |x| graph with vertex at (0, 7) d Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with vertex at (0, 7) f Positive |x| graph with v y-intercept at (0, -1) 5 a y 5 b The two graphs do not intersect, therefore there are no solutions to the equation |6 - x| = 12 x - 5. 10 Value for x cannot be negative as it equals a modulus. 11 a O b x < -13 and x > 1 12 - 23 < x < 53 13 a k = -3 b Solution is x = 6. Challenge a y f(x) = $x^2 + 9x + 8 g(x) = |4 - x| h(x) = 5 O b a x = 2$ and x = - 43 b x = 7 or x = 3 c No solution d x = 1 and x = - 17 f x = 24 or x = -12 x = - 25 or x = 2 a x g(x) = 1 - x x 8 3 b x = - 23 and x = 67 Exercise 2B 1 a i y 0 8 x There are 4 solutions: $x = -4 \pm \sqrt{7345612172227}$ ii one-to-one iii {f(x) = 12, 17, 22, 27} b x = - 43 x = -3, x = 4 x y = 2x - 9 0 e y = - | 3x + 4 | y 3246x610 c 18 Exercise 2A b 0.28 y = $12 x - 56 x - 14 y = |6 - x| ci - 1107417 bi - 33 - 32 - 1101 - 2 - 3 ii many-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii
{f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, -2, 1, 6} ii one-to-one iii {f(x) = -3, -2, -2, -2, -2} ii one-to-one iii {f(x) = -3, -2, -2, -2} ii one-to-one iii {f(x) = -3, -2, -2, -2} ii one-to-one iii {f(x) = -3, -2} ii$ function one to many ii not a function one to one not valid at the asymptote, so not a function. many to one ii function b $\pm 2\sqrt{5}$ c 4 d 2, -312345 f y i 6 y y a i f(x) = x + 9 4 f(+ 2 c i 1 ii 109 d a = -86 or a = 9 2 ii one-to-one 3.14 3.37 4 5.72 10.39 x 1 a g(x) is not a function because it is not defined for x=4 b 2 3 1 2 2 5 ii $f(x) \in \mathbb{R}$, f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 9 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii f(x) > 1 iii one-to-one f(x) = x2 + 5 ii fone 4 x b a = -3.91 or a = 3.58 9 a v 27 14 (2, 9) x O -10 c i v 2 v 1 f(x) = x + 2 ii f(x) > 0 iii one-to-one 2 3 O x 2 O 6 x b Range {2 < h(x) < 27} c a = -9, a = 0 44 11 a = 2, b = -1 12 a = 11 10 c = _25, d = __5 Exercise 2C x O d i ii 0 < f(x) < 2 iii many-to-one f(x) = 2 sin x -4 4 5 a 7 b __94 or 2.25 c 0.25 d -47 1 a 4x2 - 15 b 16x2 + 8x - 3 c __2 - 4 x = 0 44 11 a = 2, b = -1 12 a = 11 10 c = _25, d = __5 Exercise 2C x O d i ii 0 < f(x) < 2 iii many-to-one f(x) = 2 sin x -4 4 5 a 7 b __94 or 2.25 c 0.25 d -47 1 a 4x2 - 15 b 16x2 + 8x - 3 c __2 - 4 x = 0 44 11 a = 2, b = -1 12 a = 11 10 c = __25, d = __5 Exercise 2C x O d i ii 0 < f(x) < 2 iii many-to-one f(x) = 2 sin x -4 4 5 a 7 b __94 or 2.25 c 0.25 d -47 1 a 4x2 - 15 b 16x2 + 8x - 3 c __2 - 4 x = 0 44 11 a = 2, b = -1 12 a = 11 10 c = __25, d = __5 Exercise 2C x O d i ii 0 < f(x) < 2 iii many-to-one f(x) = 2 sin x -4 4 5 a 7 b __94 or 2.25 c 0.25 d -47 1 a 4x2 - 15 b 16x2 + 8x - 3 c __2 - 4 x = 0 44 11 a = 2, b = -1 12 a = 11 10 c = __25 i d = -1 12 a = 11 10 c = __25 i d = __2 bx = 94x - 21313a23bx = and x = 75e - 26369369 Answers $6789x + 111af2(x) = f() = x + 1x + 21 + 1(x + 1)x + 2bf3(x) = 2x + 33ln() 7b2x + 3c = a2x + 3ln(2)3af - 1(x) = 10 - x, {x \in \mathbb{R}} 3ch - 1(x) = (x \neq 0)$ 4 d + 1 e 16x + 5 x a fg(x) = $3x^2 - 2bx = 14x - 5agp(x) =$ x Domain becomes x < 4 4 a i 2 g(x) < _13 iii {x $\in \mathbb{R}$, x 3 iv y a 20x b x20 3 a (x + 3) - 1, qp(x) > -1 b 999 c x = -8 g - 1(x) = 1 x _ $\sqrt{6 10 3 \pm }$ 2 3 11 a -8 < x < 12 b 6 Exercise 2D 1 y g(x) > -1 iv f (x) = 2x - 1 f - 1(x) = x - 3 2 3 - 32 0 - 32 g - 1(x) = x + 1 2 x 3 x - 1 0 - 1 b i {y $\in \mathbb{R}$ } ii f -1(x) = 2x - 5 iii Domain: {x $\in \mathbb{R}$ }, Range: {y $\in \mathbb{R}$ $x \text{ iii} \{x \in \mathbb{R}, x > 0\}, g - 1(x) > 2 \text{ cif}(x) = x + 5 2 5 2 - 5 g(x) > 0 \text{ iv } y g(x) = 3 x - 2 x 5 2 - 5 g - 1(x) = 2x + 3 x 4 - x \text{ ii } f - 1(x) = 3 \text{ iii Domain: } \{x \in \mathbb{R}\}, \text{ Range: } \{y \in \mathbb{R}\} \text{ ci } x 3 1 3 x + 1 \text{ ii } g - 1(x) = 0 \{y \in \mathbb{R}\} \text{ iv } 0 \text{ bi } x - 3 \text{ ii } f - 1(x) = 2 \text{ iii } \{x \in \mathbb{R}, x > -1\} g - 1(x) > 0 \{y \in \mathbb{R}\} \text{ iv } 0 \text{ bi } x - 3 \text{ ii } f - 1(x) = 2 \text{ iii } 0 \text{ control } x + 5 2 5 2 - 5 g(x) > 0 \text{ iv } y g(x) = 3 x - 2 x 5 2 - 5 g - 1(x) = 2 x + 3 x 4 - x \text{ ii } f - 1(x) = 2 \text{ iii } 0 \text{ control } x + 5 2 5 2 - 5 g(x) > 0 \text{ iv } y g(x) = 3 x - 2 x 5 2 - 5 g - 1(x) = 2 x + 3 x 4 - x \text{ ii } f - 1(x) = 2 \text{ iii } 0 \text{ control } x + 5 2 5 2 - 5 g(x) > 0 \text{ iv } y g(x) = 3 x - 2 x 5 2 - 5 g - 1(x) = 2 x + 3 x 4 - x \text{ ii } f - 1(x) = 2 \text{ iii } 0 \text{ control } x + 5 2 5 2 - 5 g(x) > 0 \text{ iv } y g(x) = 3 x - 2 x 5 2 - 5 g - 1(x) = 2 x + 3 x 4 - x \text{ ii } f - 1(x) = 2 \text{ iii } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 - 5 g(x) = 0 \text{ in } 0 \text{ control } x + 5 2 5 2 \text{ control } x + 5 2 5 2 \text{ control } x + 5 2 5 2 \text{ control } x + 5 2 5 2 \text{ control } x + 5 2 \text{ control$ \mathbb{R} iv f -1(x) = 2x - 5 y O 2x + 3 ii g-1(x) = $\in \mathbb{R}$ a i g (x) = 1 x 1 3 c 10.5 {y $\in \mathbb{R}$ } iv f (x) = 4 - 3x y f -1(x) = 4 - x 3 4 d i g(x) > 2 ii g (x) = x2 + 3 iii {x $\in \mathbb{R}$, x > 2}, g -1(x) > 7 - 1 iv y g -1(x) = x2 + 3 4 3 0 4 3 x 4 3 y f (x) = x3 - 7 f (x) = -7 0 3 x 7 \mathbb{R} , x > 2, g -1(x) > 7 - 1 iv y g -1(x) = x2 + 3 4 3 0 4 3 x 4 3 y f (x) = x3 - 7 f (x) = -7 0 3 x 7 \mathbb{R} , x > 6, g -1(x) > 2 - 7 370370 Full worked solutions are available in SolutionBank. Online Answers Exercise 2E iv y g (x) = x2 - 7x - 8 y b 8 6 - 1 0 g - 1(x) = x - 2 2 2 3 g (x) = x3 - 8 g - 1 (x) = 3 - 8 y = f(|x|) 0 - 8 x 8 a x 2 $f -1(x) = \sqrt{x + 4 + 3}$ $a 5 b m -1(x) = \sqrt{x + 4 + 3}$ $a 5 b m -1(x) = \sqrt{x - 5 - 2} a tends to infinity b 7 2x + 1 c h - 1(x) = \frac{1}{2} + 2 - 1 c h + 1 c$ $\{x \in \mathbb{R}, x \neq 2\} x - 2 = 9xx + 82 \text{ Oyc} = 288x6 \text{ i } g(x) > 0 \text{ ii } g - 1(x) = \sqrt{x} + 8 \text{ iii } \{x \in \mathbb{R}, x > 0\}, g - 1(x) > 2 \text{ iv } y 5 67 - 10x8 - 8 \text{ Of } y = |f(x)| - 360 x y 1 y = |f(x)| - 360 x y | - 360 x y |$ $st(x) = =x = x, ts(x) = 3-x 3 _ +1 x x+1 \sqrt{-360 c -360}$ $(x + 5)(x - 4)(x + 5)(x - 4)(x + 5)(x - 4) + y c x 2 y = f(|x|) 6 1 = x+5 b \{y \in \mathbb{R}, y < 19\} 1 c f - 1 : x \rightarrow -5$. Domain is $\{x \in \mathbb{R}, x < 19 \text{ and } x \neq 0\} x O - 3 3(x + 2) 2 - 12 a f(x) = 12$ $x^{2} + x - 20x - 4 - 2 - 1012x 371371$ Answers 4 a v k(x) = b a e x = 1.95 3(x + 2) 2(x + 5) x - 4 = $a > 0 x^2 y y = |k(x)| x O x O y c b$ Both these graphs would match the original graph. y = k(|x|) y c O x m(x) = x O a, a = 0 - 3 c O x y x y = f(|x|) 1 O 372372 1 - 10 x Full worked solutions are available in SolutionBank. Online Answers Exercise 2F 1 a y b (3, 14) x O O (-3, -1) (-2, -9) (2, -9) a y 2 x O x (32, -4) e y f (3, 4) (-3, 4) 2 a (1, 12) y (-2, 2) 3 - 4O(90, 1) x x - 3 x iii A(-90, 1) x O - 180 1 360 x B(90, 1) y = 12 |h(-x)| 180 x a Range f(x) > -3 y y = 4|x| - 3 - 34 O x O y = 14 h(12 x) Exercise 2G x O A = (-2, 5), x = 0, y = 5 d y A = (0, 2), x = 2, y = -1 c y y 1 180 270 x - 1 x O B(180, 12) O(0, 0) 90 - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) 90 - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) 90 - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) 90 - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8)
y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) - 90 y b y 180 O(0, 0) - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) - 90 y b y 180 O(0, 0) - 90 y b y 180 O(0, 0) - 360 - 180 O - 12 A(-180, -12) (-2, -8) y 6 a O 1 2 x - 1 O y y d 1 e O 90 - 1 A(0, -12) O(0, 0) - 90 y b y 180 O(0, 01) ii y c 3 -90 O B(180, 3) y = h(x - 90) + 1 y 3 2 O 180 x 90 b A(-90, -2) and B(90, 2) c i y x O b y = h(x) - 2 A(-90, -2) (2, 2) x B(90, 2) O(0, 0) O -90 -180 y (-2, 2) O x 5 -5 5 (-1, 0) iii (2, 9) y = g(|x|) O y d (-1, 2) ii (1, -9) y - 5 (2, -5) (0, -7) y (2, 2) c x O (5, -1) (0, 2) (-2, -4) b i (6, -18) c y x 3 4 -3 b Range f(x) > -1 O A = (-1, 0), x = 1, y = 0 4 a A = (-1, 0), x = (0, 1), x = 2, x = -2, y = 0, y = 0, y = 0, y = 0, y = 13 and x = -3 Maximum point of f(x) is (3, 10). Minimum point of g(x) = -2|x - 1| + 6 - 2 - 0, y = 0, y =v = -52 |x| + 4 O - 85 8 5 x x O 2 v a, b v = 2|x + 4| - 5 x O 3 2 11 k > - 4 3 40 24 x = - and x = 19 21 4 a v a, b x O a 5 v 6 v = 4|x + 6| + 1 O 5 a (-2, 2) x O v O x a c 10 a c 374374 a = 10 b P(-3, 10) and O(2, 0) x = - 67 and x = -635 m(x) < 7 b x = - and x = -523 k 7 g(x) = -523 4 y = x 22 b g(x) < 7 c x = -23 and x = 7 k < 14 b = 2 a h(x) > -7 b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function. 23 c - 12 < x < 52 d k < -39 x (1, -1) b 12 and 1 12 y = -52 |x - 2| + 7678 b The graphs do not intersect, so there are no solutions. a i one-to-many ii not a function. $bx = \frac{4}{4} y = x^2 2 b g(x) < 7 c x = -\frac{23}{3} a x = \frac{7}{k} < 14 b = 2 a h(x) > -7 b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function. <math>23 c - \frac{12}{12} < x < \frac{52}{52} d k < -\frac{3}{2} 3 g(1, -1) f(x) = \frac{12}{2} - \frac{12}{52} (x - \frac{1}{2} - \frac{7}{67} 8 b The graphs do not intersect, so there are no solutions, a i one-to-many ii not a function i one-to-one ii function i many-to-one ii function i many-to-one, therefore the inverse is one-to-one ii function for a suitable domain a y (6, 4) 16 48 c x = -\frac{1}{3} 3 g(1, -1) f(x) = \frac{12}{2} x - \frac{12}{2} (x - 7 - 7 b f(x) = \frac{1$ $\frac{2}{2} - \frac{1}{4} 3 p = -4, q = 7 a x 2 = x 1 (p - 3 x 1) = 2(p - 12)(-3(p - 12) - (-5p - 43) = -1(p - 2 + 132) - 432 + 128 + 105 + 25 + 44 + 105 q = 10 + 124 + 156 + 105 q = 112 + 124 + 156 + 105 q = 112 + 124 + 156 + 105 q = 112 + 124 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 1156 + 1156 + 1125 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 125 + 124 + 124 + 125 + 124 + 124 + 124 + 124 + 125 + 124 + 124 + 125 + 124 + 124 + 125 + 124 + 124 + 125 + 124 + 124 + 124 + 125 + 124 + 124 + 125 + 124 + 125 + 124 + 12$ $\begin{array}{r} 1634k (1 + 2x + 4x^2 + 8x^3) | 1| x^2 + 2x^2 + 3x^2 | 1| x^2 - 1| x^2 - 1| x^2 - 2x^2 | x^2 - 1| x^2 - 2x^2 + 4x^2 + 4x^2 + 6x^2 - 1| x^2 - 2x^2 + 1| x^2 - 1| x^2 + 4x^2 - 8x^2 + 2x^2 + 1| x^2 - 1| x^2 - 2x^2 + x^2 - 1| x^2 - 2x^2 + 1| x^2 - 1| x^2 - 2x^2 + 2x^2 + 1| x^2 - 1| x^2 - 2x^2 + 2x^2 + 1| x^2 - 2x^2 + 2x^2 + 2x^2 - 2x^2 + 2x^2 + 1| x^2 - 2x^2 + 2x^2 + 2x^2 + 2x^2 - 2x^2 + 2x^2 + 2x^2 + 2x^2 - 2x^2 + 2x^2$ $= -1, c = -\frac{8 4 a A = 1, B = 2 b - x + 11x 2 - ... 3 1 a A = - - B = -b - 1 - x + x3 4 + ... 2 25 25 25 a A = 2, B = 10, C = 1 b - - x + x2 4 + 10 = 27 9 a A = 2, B = 5, C = -2 b 2 + 5(4 + x) - 1 - 2(3 + 2x) - 1x - 12 5 2 = 2 + (1 + -) - (1 + x) 4 (3 3 x x 2 5 2 4 = 2 + (1 - +) - (1 + x) 4 (3 3 x x 2 5 2 4 = 2 + (1 - +) - (1 + x) 4 (3 3 x x 2 5 2 4 = 2 + (1 + -) - (1 + x) 4 (3 3 x x 2 5 2 4 = 2 + (1 + -) - (1 + x) 4 (3 3 x x 2 5 2 4 = 2 + (1 + -) - (1 + x) 4 (3 3 x x 2 5 2 4 = 2 + (1 + -) - (1 + x) 4 (4 3 x x 2 + 2 + 2) + (1 + -) - (1 + x) 4 (4 + 2) + (1 + -) + (1 + x) 4 + (1 + - - - -) + (1 + x) 4 (1 + 2 + 2) + (1 + -) + (1 + x) 4 + (1 + - - - - -) + (1 + x) 4 (1 + 2) + (1 + x) 4 + (1 + x) +$ $2n 4 4 4 c r d 0.440, 2./0, 3.58, 5.48 a r b 0.501, 2.64, 3.64, 5./6 c No solutions d 1.103, 5.180 11n r __7n 11n 19n 23n 31n 35n _, a b _, , , , , 36 18 18 18 18 18 18 18 n m __a - , 0.412, 2.73 b 0, 0.644, n, 5.64, 2n 4 4 0.3, 0.5, 2.6, 2.9, 0.7, 2.4, 3.9, 5.5 8 in 2x + 4 sin x + 20 = 4 8 in 2x + 4 sin x - 24 = 0 2 sin 2x + sin x + 6 = 0 c t x - 3 in 3 3 0 30 _ = a __ = __ 6 sin 40 0 × 40 40 2 40 b 1 b 7 a b 8 a c 1 0 0 1 _ - 1 _ cos 0 - 1 __ 220 _ ≈ = __ - 2 b 3 4 5 d 1.105 < - 4.128 b 16 c 1 : 3.91 a Area of shape X = 2d 2 + _12 d 2n 2 d 2 + _12 d 2n 2 d$ as O2 is on the circumference of C1 and hence is a radius (and vice versa). Therefore, π O1AO2 is an equilateral triangle $\Rightarrow 4$ AO1O2 = _____. $3 \pi \pi \pi 2\pi$ By symmetry, 4 BO1O2 is __ $\Rightarrow 4 AO1B = __+ __= ___. <math>3 3 3 3 16\pi$ cm c 177 cm2 Student has used an angle measured in degrees – it needs to be measured in radians to use that formula. 5π __ cm2 4 $-_14 b \theta+1 (2\theta) 27 + 2(1 - __) 27 + 2 \cos 2\theta 16 a$ \approx _______ $\tan 2\theta + 3 2\theta + 3 4\theta 27 + 2(1 - __) 29 - 4\theta 2$ = = $2\theta + 3 2\theta + 3 (0) (0) 3 + 2\theta 3 - 2\theta =$ = $3 - 2\theta 2\theta + 3 b 3 17 a 32 \cos 5\theta + 203 \tan 10\theta = 182 (5\theta) 2 32(1 - __) + 203 (10\theta) = 182 2 32 - 16(25\theta2) + 2030\theta = 182 0 =$ $400\theta2 - 2030\theta + 150 0 = 40\theta2 - 203\theta + 153 b 5$, ______ 40 3 c 5 is not valid as it is not "small". ______ is "small" so is 40 valid. $\theta 2 18 1 - ______
2 \pi 3\pi 19 a 0.730$, $2.41 b - _______ 4 4 \pi 5\pi c$ _______ d - 2.48, -0.667 4 4 20 a 1 _______ Cosine 23 a Cosine can be negative so do not reject - _______ $\sqrt{2}$ squared cannot be negative but the student has 1_____ already square rooted it so no need to reject - _____. $\sqrt{2}$ b Rearranged incorrectly - square rooted incorrectly $3\pi \pi \pi 3\pi c - ___, -__, ____. 444424$ a Not found all the solutions b $\overline{0.595}$, 2.17, 3.74, $\overline{5.3125}$ a $5 \sin x = 1 + 2(1 - \sin 2x) \Rightarrow 2 \sin 2x + 5 \sin x - 3 = 0 \pi 5\pi b$. $-3 = 0 \pi 5\pi b$. $-3 = 0 \pi 5\pi b$. $= 0 \Rightarrow 4 \cos 2x - 9 \cos x + 2 = 0 b 1.3$, 5.0, 7.6, 11.2 sin 2x 27 a tan 2x = 5 sin 2x \Rightarrow = 5 sin 2x \Rightarrow = 5 sin 2x \Rightarrow cos 2x (1 - 5 cos 2x) sin 2x = 0 \pi b 0, 0.7, _, 2.5, $\pi 2 28 a y x \sqrt{3 \pi 4 \pi b} (0, _) _, 0$, (__, 0) 2 (3) 3 Challenge a $\theta = 29$ or $\theta = -3 \theta = 29$ is small, so this value is valid. $\overline{\theta} = -3$ is not small so this value is not valid. Small in this context is sec θ and $y = -\cos \theta$. As they do not meet, there are no solutions. a $y y = \cot \theta$ Therefore AP = $\cot \theta$. Exercise 6B 1 a i y - 540 - 450 - 270 1 0 - 90 - 1 ii $y 0 y = \sec \theta 540 90 270 450 \theta y = \csc \theta 540 90 270 450 \theta y = \sin 2\theta 270 360 \theta y = \sin$ 30°) 1 0 90 180 270 (120°, 1) 0 30° -1 360° 210° 300° 360° θ (0, -2) (300° , -1) b cot(90 + θ) = $-\tan \theta$ 6 7 π The graph of y = tan θ + ____ is the same as that (2) π of y = tan θ translated by ____ to the left. 2 ii The graph of y = cot($-\theta$) is the same as that of y = cot θ reflected in the y-axis. π iii The graph of y = cosec θ + ____ is the same as (4) π that of y = cot($-\theta$) is the same as that of y = cot θ reflected in the y-axis. π iii The graph of y = cosec θ + ____ is the same as (4) π that of y = cot($-\theta$) is the same as that of y = cot θ reflected in the y-axis. π iii The graph of y = cosec θ + ____ is the same as (4) π that of y = cot($-\theta$) is the same as that of y = cot θ reflected in the y-axis. π iii The graph of y = cosec θ + ____ is the same as (4) π that of y = cot($-\theta$) is the same as that of y = cot($-\theta$ is the same as that of y = cot($-\theta$) is the same as that of y = cot($-\theta$ is the same as that of y = cot($-\theta$ i cosec θ translated by __ to the left. 4 π iv The graph of y = sec θ - __ is the same as that (4) π of y = sec θ translated by __ to the right. 4 π π π b tan θ + __ = cot(-θ); cosec θ + __ = sec θ - __ (((2) 4) 4) a i a y = sec 2θ y 1 0 - 1 e y y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (135°, 0) (315°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (45°, 0) (315°, 0) (315°, 0) (315°, 0) y = 2 sec (θ - 60°) 2 (60°, 2) O f O b θ (270°, 1) 90° 180° 270° g 360° θ (90°, -1) (225°, 0) (315°, 0) 1 + sec θ 2 y 1 y 0 -2 150° 330° (15°, -1) (285°, -1) y = cosec (2 θ + 60°) 1 c (195°, 1) y = -cosec θ y 0 -1 -1 (270°, -1) θ y 1 (15°, 1) 45° 90° 135° 180° 270° 360° θ -1 180° 90° 270° 360° θ y = -cot 2 θ (180°, 0) 388388 Full worked solutions are available in SolutionBank. $\frac{1}{1 + \sec \theta + 2} = \frac{1}{2 + \csc \theta + 2} = \frac{1$ $\sin 2 A \cos 2 A (\sin 2 A \equiv \cos 2 A - 1 = \cot 2 A = R.H.S. \cos A 1 1 L.H.S. \equiv -\cos 2 A \equiv -1 1 1 b x^2 + 2 + 2 = (x +) = (2 \sec \theta)^2 = 4 \sec 2\theta x x 12 p = 2(1 + \tan 2\theta) - \tan 2\theta = 2 + \tan 2\theta 1 \Rightarrow \tan 2\theta = p - 2 \Rightarrow \cot 2\theta = -p - 2 \Rightarrow \cot 2$ $\frac{1}{6E 1 \sin \theta \text{ R.H.S.}} = \frac{1}{4 2 \cos 2\theta} = \frac{1}{1 \cos 2\theta} = \frac{1}{2} \cos 2\theta = 1 - \cos \theta = \text{L.H.S.} \sin 2\theta + 1 - \sin 2\theta = 2(1 + \sin 2\theta) - \sin 2\theta = 1 - 2 \sin 2\theta$ $\frac{1}{(x + 3) + 9} = \frac{-12 \times 21 \times 2}{(x + 7) + 9} + \frac{2}{(y + 7) + 9} + \frac{2}{(x + 7) + 9} + \frac{2}{(x + 2) + 9} + \frac{-12 \times 21 \times 2}{(x + 7) + 11 + 12 \times 2} + \frac{1}{(x + 2) + 2} + \frac{1}{(x + 2) + 9} + \frac{-12 \times 21 \times 2}{(x + 7) + 11 + 12 \times 2} + \frac{1}{(x + 2) + 2} + \frac{1}{(x + 2) + 9} + \frac{-12 \times 21 \times 2}{(x + 7) + 11 + 12 \times 2} + \frac{1}{(x + 2) + 2} + \frac{1}{(x +$ $\frac{1}{2} = \frac{1}{2} + \frac{1$ negative y and negative x quadrant. Exercise 8D 1 a (11, 0) d (1, 0), (2, 0) b e 2 a (0, -5) b 3 6 7 8 9 d (0, 12) 4 e 4 4 (7, 0) 9 (5, 0) (0, 16) 9 c (1, 0), (9, 0) c (12, -2), (32, -6) Challenge (1, 1), (e, 2) Exercise 8E 1 c (0, 0), (0, 12) (0, 1) 5 (12, 32) 25 t = 52, t = -32; (1, 0), (94, -3) 4 (1, 2), (1, -2), (4, 4), (4, -4) \pi 2 a (-1, -1), (1, -2), (1, -2), (2, 0), (0, cos 1) 4 $\sqrt{3}\sqrt{3}$ b (- __, 0), (__, 0), (0, 3), (0, 1), (0, -1) 2 2 c (1, 0) a (e + 5, 0) b (ln 8, 0), (0, -63) 2 3 c (_54, 0) b 4 sin 2t + 2 = 4 \Rightarrow 4 sin 2t = 2 \Rightarrow sin 2t = 0.5 π 5 π π 5 π 2t = _, __ \Rightarrow t = __ \Rightarrow t = _, __ \Rightarrow t = __ \Rightarrow $= 0 \text{ Discriminant} = 02 - 4 \times 2 \times (k - 2) = 0 - 8(k - 2) = 0 \Rightarrow k = 2 \text{ Therefore, } y = k \text{ is a tangent to the curve
when } k = 2. 2 - 11 \text{ a } A(4, 1), B(9, 2) \text{ b Gradient of } 1 = \underbrace{-9 - 45 \text{ c x}}_{-5y + 1} = 0 \text{ y} + \sqrt{3} \text{ x} - \sqrt{3} = 0 \text{ a } A(0, -3), B(\underline{3}4, 0) \text{ b Gradient of } 1 = 4 \text{ Equation of } 12 \text{ and } 13: y = 4x + c 4(t - 1) \text{ t} - 4 = \underbrace{-4t = 4t - 4t = 4t - 4t + ct \text{ t} + ct + 2t - (8 + c)t + 4 = 0 \text{ Tangent when discriminant} = 0 (-(8+c))2 - 4 \times 1 \times 4 = 0 \text{ 64} + 16c + c2 - 16 = 0 \text{ c} 2 + 16c + 48 = 0 (c + 4)(c + 12) = 0 \Rightarrow c = -4 \text{ or } c = -12 \text{ So, two possible equations for } 12 \text{ and } 13 \text{ are } y = 4x - 4 \text{ and } y = 4x - 4$ $-3.2 \Rightarrow y = -$ x 0.9 0.9 9 which is in the form, y = mx + c and is therefore a straight line. d 3.32 ms - 1 a 3000 m b Initial point is when t = 0. For t > 330, y is negative ie. the plane is underground or below sea level. c 26373 m a 35.3 m b Between 1.75 and 1.88 seconds (3 s.f.) c 30.3 m (3 s.f.) 100 200 a seconds b m 49 49 401401 Answers x $x 2 x 49 2 c t = \Rightarrow y = -4.9() + 10() = -x + 5x 40 2 2 2$ Therefore, the dolphin's path is a quadratic curve 250 d m 49 y - 12 x 5 a sin t = ____, cos t = _____ 12 - 12 2 2 y - 12 x _____ $22(12) + (-12) = 1 \Rightarrow x + (y - 12) = 144$ Therefore, motion is a circle with centre (0,12) and radius $\sqrt{144} = 12$. b 24 m c 2 π seconds, 12 m/s 8458.56 m 17 a 1022 m ___ b 1000 = $50\sqrt{2}$ t \Rightarrow t = $10\sqrt{2}$ $\sqrt{2}$ Sub into y = $1.5 - 4.9(10\sqrt{2})$ y = 21.5 m 21.5 > 10, therefore, the arrow will be too high c 12 m 18 a 976 m, 2 hours b 600 m 19 a 10 m b 80 m 20 a 10 m b 1 minute c 0.9 m Challenge a k = __32 2 \pi 2 \pi 2 \pi x 3 \pi 2 6 7 8 5 \pi 11 \pi b y-axis at (0, 0.5). x-axis at (___, 0) and (____, 0) 6 6 c x = 2.89, x = 5.49 a 1.287 radians b 6.44 cm $12 + 2\pi$ cm a $_12$ (r + 10) $2\theta - __12$ r $2\theta = 40 \Rightarrow 20r\theta + 100\theta = 80$ 4 \Rightarrow r = $_- - 5\theta$ b 28 cm a 6 cm b 66.7 cm2 a 119.7 cm2 b 40.3 cm Split each half of the rectangle r as shown. 2 π Area S = $_-$ r 2 12 $_-$ T $\sqrt{3}$ Area T = $_-$ r 2 8 $_-$ 3r $\pi 1 \sqrt{8}$ 2 \Rightarrow Area R =

 $- _]r^2 2 3 12 S R U R R D C a 3 sin 2 x + 7 cos x + 3 = 3(1 - cos 2x) + 7 cos x + 3 = -3 cos 2 x + 7 cos x + 6 = 3 cos 2 x - 7 cos x - 6 b x = 2.30, 3.98 10 a For small values of <math>\theta$: sin $4\theta \approx 4\theta$, cos $4\theta \approx 1 - _ 12$ (4θ) $2 \approx 1 - 8\theta^2$, tan $3\theta \approx 3\theta$ (4θ) $2 sin 4\theta - cos 4\theta + tan 3\theta \approx 4\theta - (1 - _) + 3\theta^2 = 8\theta^2 + 7\theta - 1 b - 1 11 a y = 4 - 2cosec x$ $\frac{1}{9 - \pi \pi 0.2 \pi x \cos 2x + 1 - 2 \sin x + \sin 2x 2 - 2 \sin x}{9 - \pi \pi 0.2 \pi x \cos 2x + 1 - 2 \sin x + \sin 2x 2 - 2 \sin x} = \frac{1}{2 \cos x (1 - \sin x) \cos x (1 - \sin x) 2} = \frac{1}{2 \sec x \cos x 3\pi 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin x \cos x \cos x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 11\pi 13\pi b x} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \cos x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \cos x \cos x \sin 5\pi 1} = \frac{1}{2 \sec x \cos x \cos x \cos x \cos x \cos$ $\frac{1}{4(-4.9)(50) t} = \frac{2(-4.9) t \neq -1.54, t = 6.64s \Rightarrow k = 6.64x _ bt = 25\sqrt{3} x_x_2y = 25_-4.9_+50(25\sqrt{3})(25\sqrt{3})49x_-=x2+5018750\sqrt{3}_challenge \pi-21_:12+3\pi 2 a sin x b cos x c cosec x d cot x e tan x f sec x 3 a sin 2t + cos 2t = 122y+1x-3_-22(4) + (4) = 1 \Rightarrow (x - 3) + (y + 1) = 16 y(22 + 3, 22 - 1)(-1, -1) O x b_38(2\pi \times 4) = 3\pi CHAPTER 9 Prior knowledge 9 1 a 6x - 524 y = -6x - 190.588, 3.73 1 2 b - 2 - 2\sqrt{x x c 8x} - 16x3179 3 (0, 2), (0, _, (11.1, 0) 27) Exercise 9A 1 (3x) - 123x37 y = 3x + c \Rightarrow 8t(2t - 1) = 3(4t) + c \Rightarrow 16t 2 - 20t - c = 0.25(-20) 2 - 4(16)(-c) < 0 so 64c$ = 1 = (x - 3) + (y + 1) = 16 + (2 + 3, 2 - 1)(-1, -1) + 0 + (2 + 3, 2 - 1)(-1, -1) + 0 + (2 + 3, 2 - 2) + (16 + 1) + (16 + 1) + (2 + 3, 2 - 2) + (16 + 1) + (16 + $x + \underbrace{x + \underbrace{x + 162 648 \ln 3 648 \ln 3$ -1 As $h \rightarrow 0$, $= 2e 2 - 1 x dx dx Equation of tangent: y - e2 = (2e 2 - 1)(x - 1) \Rightarrow y = (2e 2 - 1)x - 2e 2 + 1 + e2 \Rightarrow y = (2e 2 - 1)x - e2 + 1 - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the population in the year 2000 10 a b x 2 3 5 6 7 a 34x 4 ln 3 406406 c 11 a d b 20x(3 - 2x2) - 6 d 7(6 + 2x)(6x + x2)6 1 f - 9.07 millicuries/day a P0 = 37 000, k = 1.01 b 1085 c The rate of change of the populat$ = 1, y = e2, $\frac{1}{2\sqrt{7 - x} 64(2 + 8x)3 h 18(8 - x) - 51} = \frac{1}{2} - \frac{1}{2}$ $\frac{y}{y} - \frac{y}{z} - \frac{y}{z} = 0 + 16^{-7} \frac{x}{5} (3 + 3)24 (3 + 3)24 (3 + 3)24 (3 + 3)24 (3 + 3) + 45 x (2 + 3) 24 (3 + 3) + 45 x (2 + 3) + 10 + 3 + 65 x - 3)2(5 x - 3) + 45 x (2 + 3) + 10 + 3 + 65 x - 3)2(5 x - 3) + 45 x (2 + 3) + 3 + 45 x (2 + 1) + 5 + 5 x + 5 x + 3 + 4 + 5 x
+ 5 x + 5 x$ $= 0 \ 16 \ 7 \ 6x(5x - 3)3 + 3x2[3(5x - 3)2(5)] = 6x(5x - 3)3 + 45x2(5x - 3)2 = 3x(5x - 3)2(2(5x - 3) + 15x) = 3x(5x - 3)2(2(5x - 3) + 15x) = 3x(5x - 3)2(2(5x - 6) \Rightarrow n = 2, A = 3, B = 25, C = -68 \ a \ (x + 3)(3x + 11)e3x \ b \ 85e6 \ 2 \ sin \ x - 3 \ cos \ x \ 9 \ a \ (3 \ sin \ x + 2 \ cos \ x) \ ln \ (3x) +$ $9 \sqrt{3} - 4\pi 6\sqrt{3} - 8\pi 2 \arccos y dx 1 = - 11 = 2 \arccos y \times - dy \sqrt{1 - y^2} + 2 \sqrt{1 - y^2} = 0 \sqrt{3} \sin t - \sqrt{3} \cos 2t - \sqrt{3} = 0 \sqrt{3} \sin 2t + 2 = 0 \sqrt{3} \sin 2t + 2 = 0 \sqrt{3} \pi 2\pi 2 \sin t = - 11 = 2 \operatorname{sint} t \neq - \Rightarrow t = - 11 = 2 \operatorname{sint} t \neq - \Rightarrow t = - 11 = - 11 = 2 \operatorname{sint} t \neq - \Rightarrow t = - 11 = - 11 = 2 \operatorname{sint} t \neq - \Rightarrow t = - 11 = - 11 = 2 \operatorname{sint} t \neq - \Rightarrow t = - 11 = - 11 = 2 \operatorname{sint} t \neq - \Rightarrow t = - 11$ $= \frac{1}{2} = \frac$ $f 0(x) = _ 3 \sqrt{1 - x^2} (1 - x^2) 2 f 0(x) > 0 \Rightarrow x < 0, \text{ so } f(x) \text{ concave for } x \in (-1, 0) \text{ b } f 0(x) < 0 \Rightarrow x > 0, \text{ so } f(x) \text{ convex for } x \in (0, 1) \text{ c } (0, 0) \pi 1 5 \pi 1 \text{ a } _, -_, (_, -_) (6 4) 6 4 \text{ b } (1, -1) \text{ c } (0, 0) \text{ d } (0, 0) f 9(x) = 2x + 4x \ln x = 2x(1 + 2 \ln x), f 0(x) = 6 + 4 \ln x 3 _ f 0(x) = 0 \Rightarrow 4 \ln x = -6 \Rightarrow \ln x = -32 \Rightarrow x = e - 32 \Rightarrow x =$ 23 There is one point of inflection where $x = e^{-2} 10 a (0, 2)$, point of inflection b^{-2} , (e2) 12 b -2, - 2 a (-1, -), minimum (ee) cy y = xex O 2 2 (-2, -2) (-1, -e) e x A i negative ii positive B i zero ii p there is one point of inflection at (0, tan 0) = (0, 0). dy 9 a __ = 15x(3x - 1) 4 + (3x - 1) 5 dx d 2y __ = 30(3x - 1) 4 + 180x(3x - 1) 3 dx 2 32 1 1 b (_, - __), (_, 0) 9 2187 3 d 2y 10 a Although __ 2 = 0, the sign does not change, so there dx is not a point of inflection when x = 5. b (5, 0); maximum dy 2 d 2y 2 1 11 __ = _x ln x + _x - 2, __ 2 = __ ln x + 1 3 3 dx 3 dx d 2y 3 __ 2 __ > 0 \Leftrightarrow ln x > -1 \Leftrightarrow x > e - 2 3 dx 2 7 Challenge 1 A general cubic can be written as f(x) = ax3 + bx2 + cx + d. b f 0(x) = 6ax + 2b. f $0(x) = 0 \Leftrightarrow x = -$ _____ 3a Let $\varepsilon > \mathbb{R}$, $\varepsilon > 0$: b b f'(- _____ + \varepsilon) = 6ab\varepsilon > 0, f''(- ______ - \varepsilon) = - 6ab\varepsilon < 0 3a 3a b So the sign of f 0(x) changes either side of x = - ______, and 3a this is a point of inflection. 2 a f 0(x) = 12ax2 + 6bx + 2c is quadratic, so there are at most two values of x at which f 0(x) = 0. d 2y b _____ 2 = 12ax 2 + 6bx + 2c dx Discriminant = 36b2 - 8ac < 0 \Rightarrow 3b2 < 8ac d 2y So when 3b2 < 8ac d $\frac{1}{2} = 0 \text{ int of infection 2 d 1 6 k} = 2 \text{ is quadratic, 36 there are in first when 1 6 k} = 0 \text{ int d 1 g - k} = 1 \text{ and } 2 \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis final and } = 3 \text{ bis 2} \text{ bis and } \text{ bis and } = 3 \text{ bis 2} \text{ bis and } = 3 \text{ bis 2} \text{ bis and } = 3 \text{ bis 2} \text{ bis and } \text{ bis and } = 3 \text{ bis 2} \text{ bis 2} \text{ bis 2} \text{ bis and } \text{ b$ $= \sqrt{\sin x + x \cos x} \times (dx 2\sqrt{\sin x}) 2\sqrt{\sin x}$ So $2 \sin x + x \cos x = 0 \Rightarrow 2 \sin x = -x \cos x \Rightarrow 2 \tan x = -x \therefore 2 \tan x + x = 0$ a f 9(x) = 0.5e0.5x - 2x b f 9(6) = -1.957... > 0 So there exists $p \in (6, 7)$ such that f 9(p) = 0. \therefore there is a stationary point for some x = p. (6, 7). 3 409409 Answers 9 a (8 e4 3 \pi) 7 \pi e $\frac{-1}{4} + \frac{-1}{2} + \frac{-1}{2}$ $= 5 + 2\sqrt{134 + \sqrt{135 - 2\sqrt{134 - 2\sqrt{134 - 2\sqrt{135 - 2\sqrt{134 - 2\sqrt{$ b 43, -131 + x - 2ya f 9(x) = b f 9(x) = $-7e - x + 8x + 2\sqrt{x} x + 2x 4c$ f 9(x) = $x 2 \cos x + 2x \sin x + 4 \sin x u 1 = 2$, u 2 = 2.5, u 3 = 2.9 Exercise 10A 1 2 3 4 5 6 a f(-2) = -1 < 0, f(-1) = 5 > 0. Sign change implies root. b f(3) = -2.732 < 0, f(4) = 4 > 0. Sign change implies root. c f(-0.5) = -0.125 < 0, Prior knowledge 10 1 2 3 a 3.25 b 11.24 3 5 15 f(-0.2) = 2.992 > 0. Sign change implies root. d f(1.65) = -0.294 < 0, f(1.75) = 0.195 > 0. Sign change implies root. a f(1.8) = -0.00531... < 0. Sign change implies root. b f(1.8635) = -0.00531... < 0. Sign change implies root. a h(1.4) = -0.0512... < 0, h(1.5) = 0.0739... > 0. Sign change implies root. b f(1.8645) = -0.00531... < 0. Sign change implies root. b f(1.8645) = -0.00531... < 0. Sign change implies root. a h(1.4) = -0.0512... < 0, h(1.5) = 0.0739... > 0. Sign change implies root. b f(1.8645) = -0.00531... < 0. h(1.4405) = -0.00055... < 0, h(1.4415) = 0.00069... > 0. Sign change implies root. a f(2.2) = 0.020 > 0, f(2.3) = -0.087. Sign change implies root. b f(2.2185) = -0.00041... < 0. There is a sign change in the interval 2.2185 < x < 2.2195, so $\alpha = 2.219$ correct to 3 decimal places. a f(1.5) = 16.10... > 0, f(1.6) = -32.2... < 0Sign change implies root. b There is an asymptote in the graph of y = f(x) at $\pi x = \approx 1.57$. So there is not a root in this interval. 2 y $y = ex - 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p =
3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(1) = -1, f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, $q = 4 \times O 1$ c f(2) = 0.4 d p = 3, -45.37) to 2 d.p. c a = 3, b = 9 and c = 6 d y - 0.8 - 0.9 O 3 x (1.74, -45.37) - 0.5 Exercise 10B - 1 8 y = ln x c f(x) = ln x - ex + 4. f(1.4) = 0.2812... < 0, f(1.5) = -0.0762... < 0. Sign change implies root. 10 a h9(x) = 2cos2x + 4e4x. h9(-0.9) = -0.3451... < 0. h9(-0.8) = 0.1046... > 0. Sign change implies slope changes from decreasing to increasing to in over interval, which implies turning point. b h9(-0.8235) = -0.003839.... < 0, h9(-0.8225) = 0.00074... > 0. Sign change implies α lies in the range $-0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8225$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -0.8235$, so $\alpha = -0.8235 < \alpha < -$ There are either no roots or an even numbers of roots in the interval 0.2 < x < 0.8. c f(0.3) = 0.1238... > 0, f(0.4) = -0.2026... < 0, f(0.5) = -0.2026... < 0, f(0.7) = 0.2711... > 0 d There exists at least one root in the interval 0.2 < x < 0.3, 0.3 < x < 0.4 and 0.7 < x < 0.4. Additionally x = 0.6 is a root. Therefore there are at least four roots in the interval 0.2 < x < 0.3, 0.3 < x < 0.4 and 0.7 < x < 0.4. Additionally x = 0.6 is a root. Therefore there are at least four roots in the interval 0.2 < x < 0.3, 0.3 < x < 0.4 and 0.7 < x < 0.4. the interval 0.2 < x < 0.8. a $yy = x2 1 2 3 x2 + 2 x 2 - 6x + 2 = 0 \Rightarrow 6x = x 2 + 2 \Rightarrow x = 6$ ii $x 2 - 6x + 2 = 0 \Rightarrow x - 6 +$ $3 = 5x \Rightarrow x = _ 5 \text{ b i } 5.5 \text{ (1 d.p.)} = -0.5766... > 0, \text{ f } 9(0.8) = 0.0059... > 0, \text{ f } -0.516 \text{ d } x = 0 \Rightarrow x = 6x - 1 \Rightarrow x = \sqrt{6x} - 1 \text{ c The graph shows there are two roots of f(x)} = 0 \text{ b, d } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } y = 6x - 1 \text{ y} = e - x \text{ O } x \text{ b } 0 \text{ n } y = x \text{ a } x = 1 \text{ n } (0 + x + 1) + 2 \text{ o } x \text{ a } y = 0 \text{ o } x \text{ a } x = 1 \text{ n } (0 + x + 1) + 2 \text{ o } x \text{ a } x = 1 \text{ n } (0 + x + 1) + 2 \text{ o } x \text{ a } x = 1 \text{ n } (0 + x + 1) \text{ o } x = 1 \text{ n } x = 1 \text{ n } (0 + x + 1) \text{ o } x = 1 \text{ n } x = 1 \text{ n } (0 + x + 1) \text{ o } x = 1 \text{ n } x = 1 \text{ n } (0 + x + 1) \text{ o } x = 1 \text{ n }$ $x - 2x = x = \ln (x - 1) + 1 + 1 + 2 = 0 = 0$ to take the square root of a negative number over \mathbb{R} . 1 x 4 - 3x 3 - 6 = 0 \Rightarrow x 4 - x 3 - 2 = 0 3 $3111 \Rightarrow$ x 4 - 2 = x 3 \Rightarrow x = _x 4 - 2 \Rightarrow p = _, q = -2 3 3 3 x 1 = -1.260, x 2 = -1.051, x 3 = -1.168 f(-1.1315) = -0.014... < 0, f(-1.1325) = 0.024... > 0 There is a sign change in this interval, which implies α = -1.132 correct to 3 decimal places. 2-x 3 cos (x 2) + x - 2 = 0 \Rightarrow cos (x 2) = ____ 3 1/2 2-x 2-x ____ 2 \Rightarrow x = arccos () [3 3] x 1 = 1.109, x 2 = 1.127, x 3 = 1.129 f(1.12975) = 0.0001256... < 0. There is a sign change in this interval, which implies α = 1.1298 correct to 4 decimal places. f(0.8) = 0.484..., f(0.9) = . There is a change of sign in the interval, so there must exist a root in the interval, since f is continuous over the interval. $4 \cos x 4 \cos x$ $-8x + 3 = 0 \Rightarrow 8x = +3 \sin x \sin x \cos x 3$ $+ \Rightarrow x = 2 \sin x 8 a i b d 6 | 3 4 5 \sqrt{6} \sqrt{c} x 1 = 0.8142, x 2 = 0.8470, x 3 = 0.8169 d f(0.8305) = 0.0105... > 0, f(0.8315) = -0.0047... < 0.$ There is a change of sign in the interval, so there must exist a root in the interval. 9 a e $x-1 = 15 - 2x \Rightarrow x - 1 = \ln(15 - 2x) \Rightarrow x = \ln(15$ places. 10 a A(0, 0) and B(ln 4, 0) b f 9(x) = xe x + e x - 4 = e x(x + 1) - 4 412412 2 7 a f(1) = -2, f(2) = 3. There is a sign change in the interval 1 < α < 2, so there is a root in this interval. b x 1 = 1.632 4 a f 9(x) = 2x + 2 + 6 b - 0.326 x a It's a turning point, so f 9(x) = 0, and you cannot divide by zero in the Newton-Raphson formula. b 1.247 a f(1.4) = -0.020..., f(1.5) = 0.12817... As there is a change of sign in the interval, there must be a root α in this interval, b x 1 = 1.413 c f(1.4125) = -0.00076..., f(1.4135) = 0.0008112..., a f(1.3) = -0.00076..., f(1.4135) = -0.00076..., f(1.4135) = -0.0008112..., a f(1.3) = -0.00076..., f(1.4135) = -0.00076.. 0.0032... > 0, f(0.7) = -0.0843... < 0. Sign change implies root in the interval. f(1.2) = -0.0578... < 0, f(2.5) = -0.2595... < 0. Sign change implies root in the interval. b It's a turning point, so f(y) = 0, and you cannot divide by zero in
the Newton-Raphson formula. c 2.430 a f(3.4) = 0.2645... > 0, f(3.5) = -0.3781.... < 0. Sign change implies root in the interval. 3 b f 9(x) = -2x c 3.442 3x - 4 Challenge a From the graph, f(x) > 0 for all value of x > 0. Note also that xe -x > 0 when x > 0. So the same must be true 1_{-} for x > $... < 0.21_{-}2 - x f 9(x) = e (1 - 2x 2) = 0 \Rightarrow x = <math>... < 21_{-}2 - x f 9(x) = e (1 - 2x 2) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21_{-}2 - x f 9(x) = 0 \Rightarrow x = ... < 21$ $0 \text{ for } x > \sqrt{2} f(x) x n + 1 = x n - is an increasing sequence as f 9(x) 1$. Therefore the f(x) > 0 and f 9(x) < 0, for $x > \sqrt{2}$ Newton-Raphson method will fail to converge. b -0.209 2 Exercise 10D 1 2 $\pi a = E - 0.1 \sin E$, if E is a root then $f(E) = 0.6 \pi E - 0.1 \sin E - k = 0 \Rightarrow E - 0.1 \sin E = k \Rightarrow = k.6 \text{ b } 0.5782... \text{ c } f(0.5775) = -0.00069... > 0$ 0, f(0.5785) = 0.00022 < 0. Change of sign implies root in interval [0.5775, 0.5785], so root is 0.578 to 3 d.p. a A(0, 0) and B(19, 0) ln (t + 1) 1 10 b f 9(t) = -(+) 2 t+1 2 Full worked solutions are available in SolutionBank. Online Answers 3 ln (5.8 + 1) 1 10 c f 9(5.8) = -(+) 2 t+1 2 Full worked solutions are available in SolutionBank. Online Answers 3 ln (5.8 + 1) 1 10 c f 9(5.8) = -(+) 2 t+1 2 Full worked solutions are available in SolutionBank. Online Answers 3 ln (5.8 + 1) 1 10 c f 9(5.8) = -(+) 2 t+1 2 Full worked solutionBank. Online Answers 3 ln (5.8 + 1) 1 10 c f 9(5.8) = -(+) 2 t+1 2 Full worked solutionBank. 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Online Answers 3 ln (5.8 + 1) 1 10 c f 9(5.8) = -(+) 2 t+1 2 Full +) = $0.0121... > 0.225.8 + 1 \ln (5.9 + 1)$ - f 9(5.9) = 2 2 5.9 + 1 (The sign change implies that the speed changes from increasing to decreasing, so the greatest speed of the skier lies between 5.8 and 5.9. ln (t + 1) 1 10 d f 9(t) = - 0 +) = $0.2 t+1.2 \ln(t+1) + 1.10$ $= 2 t+1 (t+1)(\ln (t+1))$ $d9(x) = -e^{-0.6x(3x^2 - 19x + 15)} 5 \text{ So a = 3, b} = -19 \text{ and } c = 15 \text{ c} - 15 \text{ e} - 0.6x \neq 0 \text{ so } d9(x) = 0 \Rightarrow 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x - - 3x = 2 \text{ arccos} [-e^{-x} - -]^2 2 4 2 4 x 4 \cos(-) + 1 2 x 1 - - -x - x = 2 \text{ arccos} (-x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 5 1 1 \Rightarrow \cos(-) = -x^2 - 3x^2 - 19x - 15 = 0 \text{ i } x 5 =$ $4 \cos(2) + 1) 2 \cos 1 = 3.393, x 2 = 3.475, x 3 = 3.489, x 4 = 3.491 d x 1 = 0.796, x 2 = 0.758, x 3 = 0.752, x 4 = 0.751 e The model does support the assumption that the crime rate was increasing. The model shows that there is a minimu point _34 of the way through 2000 and a maximum point mid-way through 2003. So, the crime rate is increasing in the interval between October 2000 and june 2003. Mixed exercise 10 1 a x 3 - 6x - 2 = 0 = x 3 = 6x + 2 b c 2 a b 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 + x = 13 i 30x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 12 i 13x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 + 15 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 3x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 13 i 13x 2 - 19x + 15 = 0 = 3x 2 = 19x - 15 = x = 12 i 10 1 0 t 7.874 (3 d, p) e Restrict the range inplies there is a root in this interval. B(0x - 0 - 1) = 0 = 0 = x 2 = 4 = x = x = 4 + 10 + 10 + 10 = 0 = 2 = 2 = 0 = x 2 = 10 + 10 + 10 = 0 = 15x + 13 = 10 = 0 = 0 = x 2 = 4 = x = x = 4 + 10 + 10 = 0 = 15x + 13 = 10 = 0 = 12x + 10 = 10 = 10x + 10 = 10x + 10 = 10x + 10 =$ change implies root. 11 0 = 6, 0 = (3 - 2x)e, 0.8x = 1 = 3 - 2x = 6 - 0.8x = 3 - 2x = 4 = -0.8x = 3 - 2x = 4 - 0.8x = 3 - 2x = -0.8x = 3 - 2x = 4 - 0.8x = 3 - 2x = -0.8x = 3 - 2x = 4 - 0.8x = 3 - 2x = -0.8x = 3 - 2x = 4 - 0.8x = 3 - 2x = -0.8x = 0 - 0.8x = 0.8x = -0.8x = 0 - 0.8x = 0.8x = -0.8x = 0.8x = -0.8x = 0 - 0.8x = 0.8x = -0.8x = 0.8x = 0.8x = -0.8x = 0.8x $\frac{1}{2} \frac{1}{2} \frac{1}$ $\frac{1}{12} - \frac{1}{12} - \frac{1}{12}$ $(3x - 5) \Rightarrow x = \ln (3x - 5) + 2b x 0 = 4, x 1 = 3.9459, x 2 = 3.9225, x 3 = 3.9121 17$ a
f(0.2) = -0.01146... < 0, f(0.3) = 0.1564... > 0. Sign change implies root. 1 1 _____ b + 4 x 2 = 0 \Rightarrow _____ 3 = -4 x 2 (x - 2) 3 (x - 2) \sqrt{21 22 23 25 27 28 29 30 b 18} a It's a turning point, so the gradient is zero, which means dividing by zero in the Newton $= 2.923\ 23.825...\ 19\ a$ i There is a sign change between f(0.2) and f(0.3). Sign change implies root. ii There is a sign change between f(2.6) and f(2.7). Sign change implies root. 2 2 3 3 3 _ 1 1 _ x - x 3 + _ - 4 = 0 = _ x 3 = x 3 - _ + 4 b x x 10 10 34 $\sqrt{35}\ 37\ 38\ 40\ 0\ y\ 0\ 0.29836\ 0.89022$ Raphson formula. -0.5515... b x n+1 = 2.9 - $1.99207 3.96243 7.38906 0.2 0.4 0.6 0.8 1 c 108 cm3 (3 s.f.) dC a = -kC, because k is the constant of dt proportionality. The negative sign and k > 0 indicates rate of decrease. c k = __14 ln 10 b C = Ae-kt k = -4, k = 16 36 130.3° 5 10 2 __i - ___j - ___k a 10i - 5j - 2k b ___ \sqrt{129} \sqrt{129} \sqrt{129} \rightarrow -c 100.1° d Not parallel: PQ \neq mAB. k = 2 39 p = -2, q = -8, r = -4 a a = 6, b = -7, c = -1 b - 6i - 2j + 4k ___ C (-3i - j + 2k) m s - 2 d \sqrt{14 m s} - 2 Challenge 1 _____ 3 10 2 2 10 ___ 1 1 \Rightarrow x 3 = __(4 + x 3 - _) \Rightarrow x = ((4 + x 3 - _)) x x 3 3 x c 2.168 (4 s.f.) d 3.35\% 2x - 1 4 - 1 _____ 31 a + ___ = (x - 1)(2x - 3) x - 1 2x - 3 A(2x - 3)2 10(2x - 3)2 b y$ $b r = t + A9 32 a 2 2 5 [] 8\pi 16\pi r dV 33 a Rate in = 20, rate out = -kV. So = 20 - kV dt 20 20 b A = and B = - kk = 3 - 1 - 1 = 2 = (x - 2) 3 = 2 + 2 = x 4x 4x c x 0 = 1, x 1 = 1.3700, 75, x 2 = 1.4893, x 3 = 1.5170, x 4 = 1.5228 d f(1.5235) = 0.0412... > 0,$ c v = (x - 1) (x - 1) 1 = 6.9 k 3 k $f(\overline{1.5245}) = -0.0050... < 0 c x 0 = 2.5, x 1 = 2.6275, x 2 = 2.6406, x 3 = 2.6419, x 4 = 2.6420 - 1.10670714 d 0.3 - 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.0050... < 0.005$ $= 0.208 - 12.02597883 \text{ a } \text{R} = 0.37, \alpha = 1.2405 2 \text{ x } \text{b } \text{v9}(\text{x}) = -0.148 \sin(2 + 1.2405) 5 \text{ c } \text{v9}(4.7) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.148 \sin(2 + 1.2405) 5 \text{ c } \text{v9}(4.7) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.148 \sin(2 + 1.2405) 5 \text{ c } \text{v9}(4.7) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.148 \sin(2 + 1.2405) 5 \text{ c } \text{v9}(4.7) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.00312... < 0, \text{v9}(4.8) = 0.002798... > 0. \text{ Sign change implies maximum or minimum. d } 12.607 \text{ e } \text{v9}(12.60665) = -0.00312... < 0, \text{v9}(4.8) = -0.00312... < 0, \text{v9}(4.8$ 0.0000037... > 0, y9(12.60675) = -0.0000022... < 0. Sign change implies maximum or minimum. a=1 a cos 7x + cos 3x = cos (5x + 2x) + cos (5x - 2x) = cos 5x cos 2x + sin 5x sin 2x + cos 5x ≡ $2 \sec t 1 =$ $h \rightarrow 0 h \rightarrow 0 h h \cos x \cos h - \sin x \sin h - \cos x$ $= \lim h \rightarrow 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos
h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \cos h - 1 \sin h = \lim h \oplus 0 h \otimes 0 h \otimes 0 h = 1 h \oplus 0 h \oplus 0 h \otimes 0 h = 1 h \oplus 0 h \oplus 0$ $\cos x - \sin x = -\sin x$ (h)] h $\rightarrow 0$ [(h a p = 4, p = -4 b Use p = -4, -18 $\overline{432}$ 49 ____ 705 ___ (-8, 64) a a u1 = a, u2 = 96 = ar, S $\infty = 600 =$ ____ 1-r 96 ___ r So = lim $= 600 \Rightarrow 96 = 600r(1 - r)$ \Rightarrow 96 = 600r 1-r - 600r 2 and therefore 25r2 - 25r + 4 = 0 b r = 0.2, 0.8 c a = 120 d n = 39 a y y = 2.79 - 0.01(x - 11) 2, A = 2.79, B = 0.01, C = -11 11 m from goal, height of 2.79 m. 27.7 m (or 27.70 m) x = 0, y = 1.58. The ball will enter the goal. Surface area of box = 2x2 + 2(2xh + xh) = 2x2 + 6xh Surface area of lid = 2x2 + 2(6x + 3x) = 2x2 + 18x Total surface area = $4x^2 + 6xh + 18x = 53565356 - 18x - 4x^2 2678 - 9x - 2x^2 Soh = 18x - 18x$ 3x 6x V = 2x2h = 23 (2678x - 9x2 - 2x3) b 6x2 + 18x - 2678 = 0, x = 19.68 d 2V d 22 648.7 cm3 c $2 < 0 \Rightarrow$ maximum e 31.7% dx 12 a b c d 13 a Exam-style practice: Paper 2 1 2 3 4 5 O -14 B9 (-6, 11) (0, 6) = A C (-10, 0) (-3, 0) O b E (4, 0) 6 7 x y B (-6, 16) A (0, 5) D9 8 E (4, 5) C (-3, 5) O D (0, -1) 9 x 10 c y C (0, 0) O A (-7, 0) D (3, 12) 11 12 13 E (7, 0) x 14 B (-3, -22) x = 0.74, x = 5.54 dV 10 a = -kV \Rightarrow ln V = -kt + c \Rightarrow V = V 0 e -kt dt 5 1 b k = ln(), V0 = £35 100 c t = 11.45 years 3 3 11 a 14.9 miles b It is unlikely that a road could be built in a straight line, so the actual length of a road will be greater than 14.9 miles. 9 422422 a = -1, b = -4, c = 4 a y = -4x + 28 b y = -14 x + -52 c R(-10, 0) d 204 units 21 () y = 3 ln x + 1, x > -1 a Student did not apply the laws of logarithms correctly in moving from the first line to the second line: $4A + B \neq 4A + 4B$ b x = -2. Note x \neq -5 y a -6 -6 10 xb - 6 < y < 18 ca = -7, a = 0 $ak = 4 bx = -2, x = 3 + \sqrt{7}, x = 3 - \sqrt{7} a Area = 12 (x - 10)(x - 3) sin 30^\circ = 14 (x - 13x + 30 = 44, and x - 13x - 14 = 0 bx = 14 (x \neq 1, as x - 10 and x - 3 must be positive.) 13\pi ah = -5, k = 2, c = 36 b$ 2 23 7 $aA = 4, B = 5, C = -6 b - x - x + 28 32 \rightarrow OA = a and x - 3 must be positive.$ $OB = b \rightarrow \rightarrow \rightarrow 1 \rightarrow 1 \rightarrow 1 MN = MB + BN = 2 OB + 2 BA = 2 b + 12 (-b + a) = 12 a \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow Therefore OA and MN are parallel and MN = 12 OA as required. a 2.46740 b 2.922 c 3.022 d 3.3% a £3550 b £40 950 c £51 159.09 a R = 0.41, \alpha = 1.3495 b 40 cm at time t = 2.70 seconds c 0.38 seconds and 5.02 seconds. a h9(t) = 3e$ $-0.3(x - 6.4) - 8e \ 0.8(x - 6.4) b$ $38 \ e -0.3(t - 6.4) = e \ \overline{0.8}(t - 6.4) = e \ \overline{0.8}(t - 6.4) = 0.8(t - 6.4) = 0.8($ which implies a turning point. Full worked solutions are available in SolutionBank. 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