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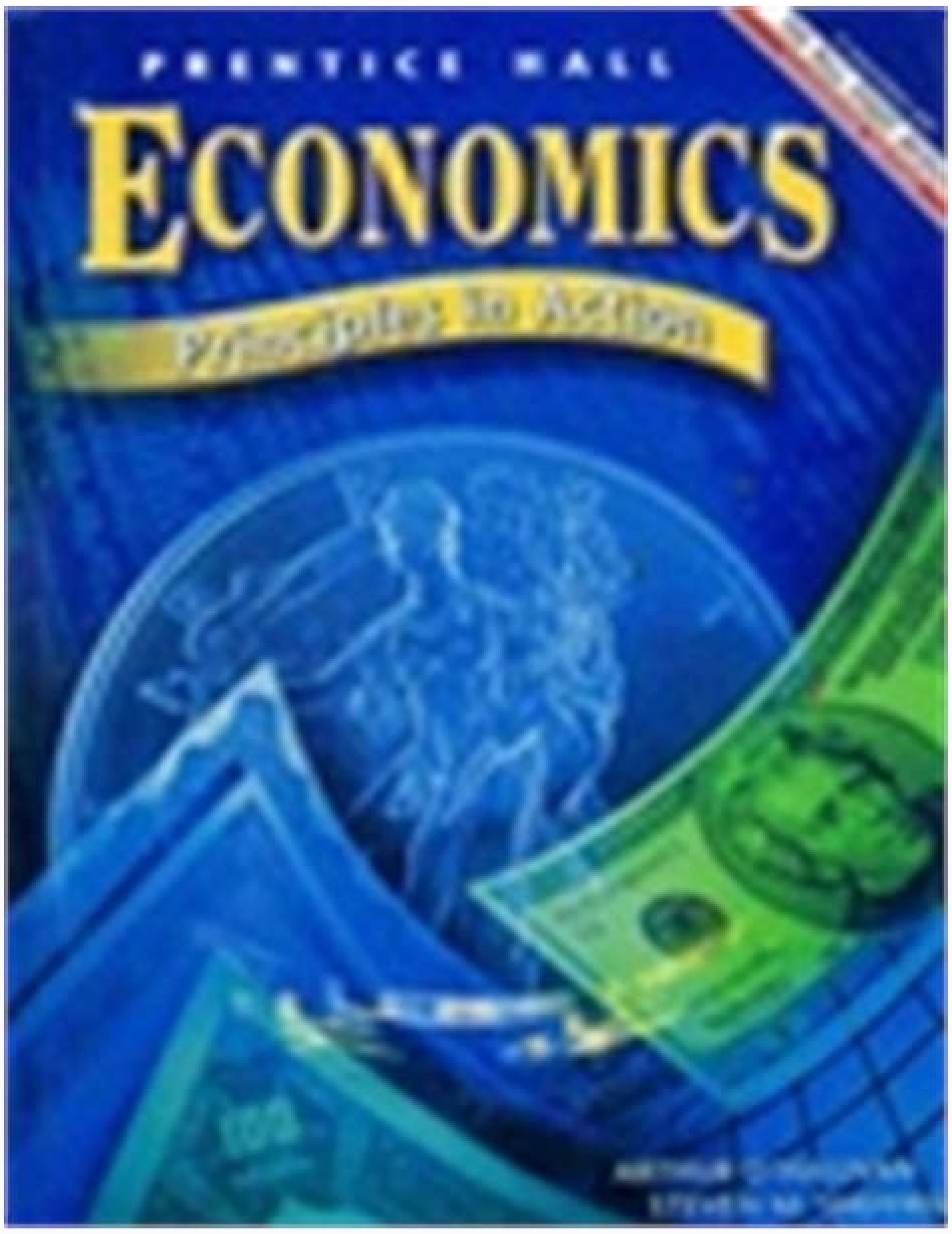


NAME ______________________ DATE __________ PERIOD ____

8-2 Practice

Factoring Using the Distributive Property

Factor each polynomial.

1. $64 - 40ab$	**2.** $4d^2 + 16$	**3.** $6r^2s - 3rs^2$
4. $15cd + 30c^2d^2$	**5.** $32a^2 + 24b^2$	**6.** $36xy^2 - 48x^2y$
7. $30x^3y + 35x^2y^2$	**8.** $9c^3d^2 - 6cd^3$	**9.** $75b^2c^3 + 60bc^3$
10. $8p^2q^2 - 24pq^3 + 16pq$	**11.** $5x^3y^2 + 10x^2y + 25x$	**12.** $9ax^3 + 18bx^2 + 24cx$
13. $x^2 + 4x + 2x + 8$	**14.** $2a^2 + 3a + 6a + 9$	**15.** $4b^2 - 12b + 2b - 6$
16. $6xy - 8x + 15y - 20$	**17.** $-6mn + 4m + 18n - 12$	**18.** $12a^2 - 15ab - 16a + 20b$

Solve each equation. Check your solutions.

19. $x(x - 32) = 0$	**20.** $4b(b + 4) = 0$	**21.** $(y - 3)(y + 2) = 0$
22. $(a + 6)(3a - 7) = 0$	**23.** $(2y + 5)(y - 4) = 0$	**24.** $(4y + 8)(3y - 4) = 0$
25. $2z^2 + 20z = 0$	**26.** $8p^2 - 4p = 0$	**27.** $9x^2 = 27x$
28. $18x^2 = 15x$	**29.** $14x^2 = -21x$	**30.** $8x^2 = -26x$

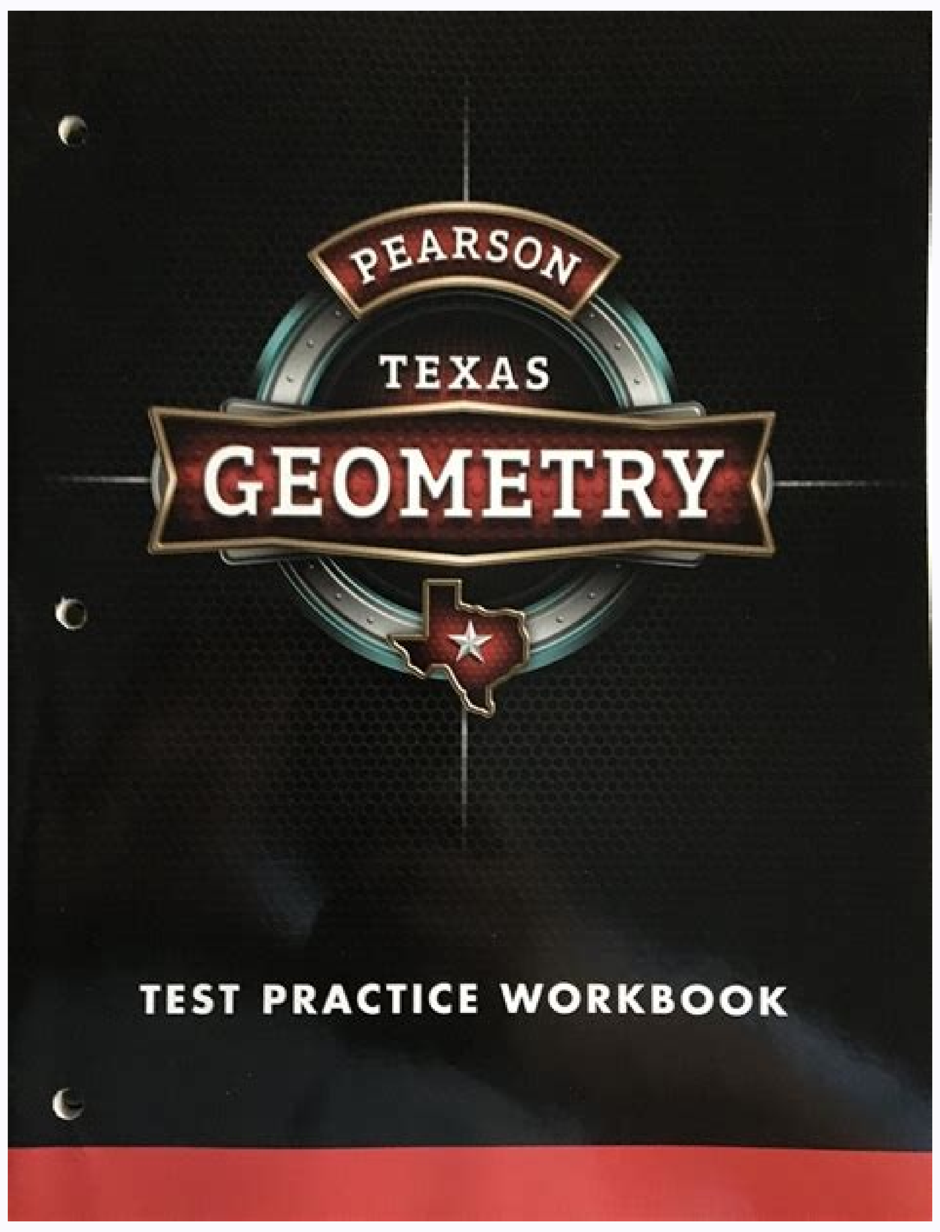
LANDSCAPING For Exercises 31 and 32, use the following information.

A landscaping company has been commissioned to design a triangular flower bed for a mall entrance. The final dimensions of the flower bed have not been determined, but the company knows that the height will be two feet less than the base. The area of the flower bed can be represented by the equation $A = \frac{1}{2}b^2 - b$.

31. Write this equation in factored form.

32. Suppose the base of the flower bed is 16 feet. What will be its area?

33. **PHYSICAL SCIENCE** Mr. Alim's science class launched a toy rocket from ground level with an initial upward velocity of 60 feet per second. The height h of the rocket in feet above the ground after t seconds is modeled by the equation $h = 60t - 16t^2$. How long was the rocket in the air before it returned to the ground?



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What is pseudocode?Starting Out With Visual Basic (8th Edition)What is a count-controlled loop?Starting Out with Python (3rd Edition)Write a program to print the value of EOF.C Programming LanguageRepeat Problem 3.84 for nitrogen gas.Fundamentals Of ThermodynamicsFind the change in length of side AB.Mechanics of Materials, 7th Edition Table of contents : Cover......Page 1Contents......Page 3Overarching themes......Page 5Extra online content......Page 7Chapter 1: Algebraic methods......Page 10 1.1: Proof by contradiction......Page 11 1.2: Algebraic fractions......Page 14 1.3: Partial fractions......Page 18 1.4: Repeated factors......Page 21 1.5: Algebraic division......Page 23 Mixed exercise: 1......Page 28Chapter 2: Functions and graphs......Page 31 2.1: The modulus function......Page 32 2.2: Functions and mappings......Page 36 2.3: Composite functions......Page 41 2.4: Inverse functions......Page 45 2.5: y = |f(x)| and y = f(|x|)......Page 49 2.6: Combining transformations......Page 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Alistair Macpherson, Bronwen Moran, Joe Petran, Keith Pledger, Cong San, Harry Smith, Geoff Staley and Dave Wilkins Contents Contents Overarching themes iv Extra online content vi Review exercise 1 107 5 Radians 113 1 Algebraic methods 1 5.1 Radian measure 114 1.1 Proof by contradiction 2 5.2 Arc length 118 1.2 Algebraic fractions 5 5.3 Areas of sectors and segments 122 1.3 Partial fractions 9 5.4 Solving trigonometric equations 128 1.4 Repeated factors 12 5.5 Small angle approximations 133 1.5 Algebraic division 14 Mixed exercise 5 135 Mixed exercise 1 19 6 Trigonometric functions 142 2 Functions and graphs 22 6.1 Secant, cosecant and cotangent 143 2.1 The modulus function 23 6.2 Graphs of sec x, cosec x and cot x 145 2.2 Functions and mappings 27 6.3 Using sec x, cosec x and cot x 149 2.3 Composite functions 32 6.4 Trigonometric identities 153 2.4 Inverse functions 36 6.5 Inverse trigonometric functions 158 2.5 y = |f(x)| and y = f(|x|) 40 Mixed exercise 6 162 2.6 Combining 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Mixed exercise 12 273 Mixed exercise 10 289 Integration 293 11.1 Integrating standard functions 294 11.2 Integrating f(ax + b) 296 349 Review exercise 3 352 Exam-style practice: Paper 1 358 Exam-style practice: Paper 2 361 Answers 365 Index 423 iii Overarching themes Overarching themes The following three overarching themes have been fully integrated throughout the Pearson Edexcel AS and A level Mathematics series, so they can be applied alongside your learning and practice. 1. Mathematical argument, language and proof • Rigorous and consistent approach throughout • Notation boxes explain key mathematical language and symbols • Dedicated sections on mathematical proof explain key principles and strategies • Opportunities to critique arguments and justify methods 2. Mathematical problem solving • Hundreds of problem-solving questions, fully integrated into the main exercises • Problem-solving boxes provide tips and strategies • Structured and unstructured questions to build confidence • Challenge boxes provide extra stretch 3. Mathematical modelling The Mathematical Problem-solving cycle specify the problem interpret results collect information process and represent information • Dedicated modelling sections in relevant topics provide plenty of practice where you need it • Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model • Dedicated chapter in Statistics & Mechanics Year 1/AS explains the principles of modelling in mechanics Finding your way around the book Access an online digital edition using the code at the front of the book. Each chapter starts with a list of objectives The Prior knowledge check helps make sure you are ready to start the chapter iv The real world applications of the maths you are about to learn are highlighted at the start of the chapter with links to relevant questions in the chapter Overarching themes Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard Problem-solving boxes provide hints, tips and strategies, and Watch out boxes highlight areas where students often lose marks in their exams Exercises are packed with exam-style questions to ensure you are ready for the exams Exam-style questions are flagged with E Problem-solving questions are flagged with P Each chapter ends with a Mixed exercise and a Summary of key points Challenge boxes give you a chance to tackle some more difficult questions Step-by-step worked examples focus on the key types of questions you'll need to tackle Each section begins with explanation and key learning points Every few chapters a Review exercise helps you consolidate your learning with lots of exam-style questions Two A level practice papers at the back of the book help you prepare for the real thing v Extra online content Extra online content Whenever you see an Online box, it means that there is extra online content available to support you. Differentiation 12A 1 a Examples of estimates of gradients: Gradient of tangent at x = −1 is y2 − y1 3 −1 = x2 − x1 (−1) − (−0.5) = − 4 Gradient of tangent at x = 0 is y2 − y1 1 − (−1) = x2 − x1 (−0.5) − (0.5) = −2 Gradient of tangent at x = 1 is y2 − y1 (−1) − (−1) = x2 − x1 2−0 =0 Gradient of tangent at x = 2 is y2 − y1 (−1) − 1 = x2 − x1 1.5 − 2.5 =2 Gradient of tangent at x = 3 is y2 − y1 3 −1 = =4 x2 − x1 3 − 2.5 SolutionBank SolutionBank provides a full worked solution for every question in the book. x-coordinate Estimate for gradient of curve Online Full worked solutions are available in SolutionBank. −1 0 1 2 3 −4 −2 0 2 4 2 c i Gradient of AD = y2 − y1 x2 − x1 0.8 − 0.19 = 0.6 − 0.9 = −1.21 (3 s.f.) ii Gradient of AC = y2 − y1 x2 − x1 0.8 − 0.6 = 0.6 − 0.8 = −1 iii Gradient of AB = y2 − y1 x2 − x1 0.8 − 0.51 = 0.6 − 0.7 = − 0.859 (3 s.f.) d As the points move closer to A, the gradient tends to − 0.75. 3 a i Gradient= 16 − 9= 7= 7 4−3 1 b The gradient of the curve at the point where x = p is 2p − 2. ii Gradient = 12.25 − 9 3.25 = = 6.5 3.5 − 3 0.5 c Gradient of tangent at x = 1.5 is y2 − y1 (−1.7) − 0.3 = x2 − x1 0.5 − 2.5 =1 2p − 2 = 2(1.5) − 2 = 1 iii Gradient = 9.61 − 9 0.61 = = 6.1 3.1 − 3 0.1 iv Gradient = 9.0601 − 9 0.0601 = = 6.01

When 3 2 dx f (x + h) − f (x) 2 h = 12 − 8 2 2 10(x + h) − 10 x © Pearson Education Ltd 2017. Copying permitted for purchasing institution only. This material is not copyright free. 3 = lim h →0 = 11 h 2 4 2 2 10 x + 20 xh + 10h − 10 x 2 dy = lim 6×3− 3 h h →0 , x 3= = When 3 dx 2 20 xh + 10h 2 = lim = 18 − h →0 h 27 h(20 x + 10h) 25 = lim h →0 = 17 h 27 = lim(20 x + 10h) 4, h →0 The gradients at points A, B and C are 25 As h → 0, 20x + 10h → 20x 11 34 and 17 27 , respectively. So f ′(x) = 20x 3 y = 7x2 − x 4 2 a A has coordinates (1, 4). dy 2 = 14x − 3x The y-coordinate of B is dx (1 + δx)3 + 3(1 + δx) 3 2 y d 3δx + 3 3 + = 1 + 3δx + 3(δx) + (δx) = 16 when 2 3 4 x d = (δx) + 3(δx) + 6δx 14x − 3x2 = 16 Gradient of AB 3x2 − 14x + 16 = 0 y −y = 2 1 (3x − 8)(x − 2) = 0 x2 − x1 8 2 3 x = or x = 2 (δx) + 3 (δx) + 6δx + 4 − 4 3 = δx 2 3 y = x3 − 11x + 1 5 (δx) + 3 (δx) + 6δx dy = = 3x2 − 11 δx dx 2 = (δx) + 3δx + 6 dy = 1 when x d 2 b As δx → 0, (δx) + 3δx + 6 → 6 3x2 − 11 = 1 at Therefore, the gradient of the curve 3x2 = 12 point A is 6. x2 = 4 x = ±2 3 1 −2 2 x = 2, y = 2 − 11(2) + 1 = −13 When 2 x + 3 + 3x = + 3 + 3 y = 3x 3 = 15 x2 When x = −2, y = (−2) − 11(−2) + 1 2 The gradient is 1 at the points (2, −13) dy −3 = 6x − 2x = 6x − 3 and (−2, 15). x dx f(x) = 10x2 f ′(x) = lim h →0 Download all the solutions as a PDF or quickly find the solution you need online When x = 1, 9 = x + 9x−1 x 9 −2 f ′(x) = 1 − 9x = 1 − x 2 2 dy = 6 ×1 − 3 1 dx =4 © Pearson Education Ltd 2017. 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If you spot an error, please do contact us at so we can make sure it is corrected. viii 1 Algebraic methods Objectives After completing this chapter you should be able to: ● Use proof by contradiction to prove true statements → pages 2–5 ● Multiply and divide two or more algebraic fractions → pages 5–7 ● Add or subtract two or more algebraic fractions → pages 7–8 ● Convert an expression with linear factors in the denominator into partial fractions → pages 9–11 ● Convert an expression with repeated linear factors in the denominator into partial fractions → pages 12–13 ● Divide algebraic expressions → pages 14–17 ● Convert an improper fraction into partial fraction form → pages 17–18 Prior knowledge check 1 Factorise each polynomial: a x2 − 6x + 5 b x2 − 16 c 9x2 − 25 You can use proof by contradiction to prove that there is an infinite number of prime numbers. Very large prime numbers are used to encode chip and pin → Example 4, page 3 transactions. ← Year 1, Section 1.3 2 Simplify fully the following algebraic fractions. x2 − 9 2__________ x 2 + 5x − 12 a _________ b x2 + 9x + 18 6x 2 − 7x − 3 2 x − x − 30 c ____________ ← Year 1, Section 7.1 −x 2 + 3x + 18 3 For any integers n and m, decide whether the following will always be odd, always be even, or could be either. a 8n b n−m c 3m d 2n – 5 ← Year 1, Section 7.6 1 Chapter 1 1.1 Proof by contradiction A contradiction is a disagreement between two statements, which means that both cannot be true. Proof by contradiction is a powerful technique. ■ To prove a statement by contradiction you start by assuming it is not true. You then use logical steps to show that this assumption leads to something impossible (either a contradiction of the assumption, or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement was true. Example Notation A statement that asserts the falsehood of another statement is called the negation of that statement. 1 Prove by contradiction that there is no greatest odd integer. Begin by assuming the original statement is false. This is the negation of the original statement. Assumption: there is a greatest odd integer, n. n + 2 is also an integer and n + 2 > n n + 2 = odd + even = odd So there exists an odd integer greater than n. This contradicts the assumption that the greatest odd integer is n. You need to use logical steps to reach a contradiction. Show all of your working. The existence of an odd integer greater than n contradicts your initial assumption. Therefore, there is no greatest odd integer. Finish your proof by concluding that the original statement must be true. Example 2 Prove by contradiction that if n2 is even, then n must be even. Assumption: there exists a number n such that n2 is even but n is odd. This is the negation of the original statement. n is odd so write n = 2k + 1 You can write any odd number in the form 2k + 1 where k is an integer. = (2k + = + 4k + 1 2 = 2(2k + 2k) + 1 So n2 is odd. This contradicts the assumption that n2 is even. n2 1)2 4k2 Therefore, if n2 is even then n must be even. All multiples of 2 are even numbers, so 1 more than a multiple of 2 is an odd number. Finish your proof by concluding that the original statement must be true. ■ A rational number can be written as __a , where a and b are integers. b ■ An irrational number cannot be expressed in the form __a , where a and b are integers. 2 b Notation ℚ is the set of all rational numbers. Algebraic methods Example 3 __ Prove by contradiction that √2 is an irrational number. __ Assumption: √2 is a rational number. __ a Then √2 = __ for some integers, a and b. b Also assume that this fraction cannot be reduced further: there are no common factors between a and b. a2 So 2 = ___2 or a2 = 2b 2 b This means that a2 must be even, so a is also even. If a is even, then it can be expressed in the form a = 2n, where n is an integer. So a2 = 2b2 becomes (2n)2 = 2b2 which means 4n2 = 2b2 or 2n2 = b2. This means that b2 must be even, so b is also even. If a and b are both even, they will have a common factor of 2. This contradicts the statement that a and b have no common factors. __ Therefore √2 is an irrational number. Example Begin by assuming the original statement is false. This is the definition of a rational number. If a and b did have a common factor you could just cancel until this fraction was in its simplest form. Square both sides and make a2 the subject. We proved this result in Example 2. Again using the result from Example 2. All even numbers are divisible by 2. Finish your proof by concluding that the original statement must be true. 4 Prove by contradiction that there are infinitely many prime numbers. Assumption: there is a finite number of prime numbers. List all the prime numbers that exist: p1, p2, p3, ..., pn Consider the number N = p1 × p2 × p3 × ... × pn + 1 When you divide N by any of the prime numbers p1, p2, p3, ..., pn you get a remainder of 1. So none of the prime numbers p1, p2, p3, ..., pn is a factor of N. So N must either be prime or have a prime factor which is not in the list of all possible prime numbers. This is a contradiction. Therefore, there is an infinite number of prime numbers. Begin by assuming the original statement is false. This is a list of all possible prime numbers. This new number is one more than the product of the existing prime numbers. This contradicts the assumption that the list p1, p2, p3, ..., pn contains all the prime numbers. Conclude your proof by stating that the original statement must be true. 3 Chapter 1 Exercise P 1A 1 Select the statement that is the negation of 'All multiples of three are even'. A All multiples of three are odd. B At least one multiple of three is odd. C No multiples of three are even. P 2 Write down the negation of each statement. a All rich people are happy. b There are no prime numbers between 10 million and 11 million. c If p and q are prime numbers then (pq + 1) is a prime number. d All numbers of the form 2n − 1 are either prime numbers or multiples of 3. e At least one of the above four statements is true. P 3 Statement: If n2 is odd then n is odd. a Write down the negation of this statement. b Prove the original statement by contradiction. P 4 Prove the following statements by contradiction. a There is no greatest even integer. b If n3 is even then n is even. c If pq is even then at least one of p and q is even. d If p + q is odd then at least one of p and q is odd. E/P 5 a Prove that if ab is an irrational number then at least one of a and b is an irrational number. (3 marks) b Prove that if a + b is an irrational number then at least one of a and b is an irrational number. (3 marks) c A student makes the following statement: If a + b is a rational number then at least one of a and b is a rational number. Show by means of a counterexample that this statement is not true. P E/P 6 Use proof by contradiction to show that there exist no integers a and b for which 21a + 14b = 1. Hint 7 a Prove by contradiction that if n2 is a multiple of 3, n is a multiple of 3. (3 marks) Hint __ b Hence prove by contradiction that √3 is an irrational number. (3 marks) 4 (1 mark) Assume the opposite is true, and then divide both sides by the highest common factor of 21 and 14. Consider numbers in the form 3k + 1 and 3k + 2. Algebraic methods P E/P E/P 8 Use proof by contradiction to prove the statement: 'There are no integer solutions to the equation x 2 − y 2 = 2' 3 Hint You can assume that x and y are positive, since (−x)2 = x2. __ 9 Prove by contradiction that √2 is irrational. (5 marks) 10 This student has attempted to use proof by contradiction to show that there is no least positive rational number: Assumption: There is a least positive rational number. Let this least positive rational number be n. a As n is rational, n = __ where a and b are integers. b a a−b n − 1 = __ − 1 = ____ b b a−b Since a and b are integers, _____ is a rational number that is less than n. b This contradicts the statement that n is the least positive rational number. Therefore, there is no least positive rational number. Problem-solving You might have to analyse student working like this in your exam. The question says, 'the error', so there should only be one error in the proof. a Identify the error in the student's proof. (1 mark) b Prove by contradiction that there is no least positive rational number. (5 marks) 1.2 Algebraic fractions Algebraic fractions work in the same way as numeric fractions. You can simplify them by cancelling common factors and finding common denominators. ■ To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators. Example 5 Simplify the following products: 3 5 a c a __ × __ b __ × __ 5 9 b a 1 3 51 1 × 1 1 a __ × __ = _____ = __ 93 1 × 3 3 15 1a c 1×c c b __ × __ = _____ = __ b a1 b × 1 b 3 3 x+1 x+1 c _____ × _____ = _____ × ____________ 2 2 (x + 1)(x −

59 A B 7 Show that ______________2 can be written in the form ____ + _____ + _______2 where x + 5 3x − 1 (3x − 1) (x + 5)(3x − 1) A, B and C are constants to be found. (4 marks) 8 Expr[illegible] algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it [illegible] denominator are equal. The degree of the numerator is greater than the degree of the denominator. Notation ■ You can either use: • algebraic division • or the relationship F(x) = [illegible] remainder can be numbers or functions of x. to convert an improper fraction into a mixed fraction. Method 1 Use algebraic long division to show that: F(x) x + 5x + 8 __________ 2 x−2 [illegible] divisor Example 11 x3 + x2 − 7 D Given that ________ ≡ Ax 2 + Bx + C + ____ , find the values of A, B, C and D. x−3 x−3 Using algebraic long division: x2 + 4x x − 3 x3 + x2 x3 − 3x2 [illegible] for in the question. Algebraic methods x3 + x2 − 7 So ___________ = x2 + 4x + 12 x−3 with a remainder of 29. x3 + x2 − 7 ___________ x−3 29 = x2 + 4x + 12 + ______ x−3 So A = 1, B [illegible] unknown asked for in the question. Example 12 Given that x 3 + x 2 − 7 ≡ (Ax 2 + Bx + C)(x − 3) + D , find the values of A, B, C and D. Let x = 3: 27 + 9 − 7 = (9A + 3B + C) × 0 + [illegible] coefficients in x3: 1=A Compare coefficients in x2: 1 = −3A + B 1 = −3 + B Therefore A = 1, B = 4, C = 12 and D = 29 and we can write x 3 + x 2 − 7 ≡ (x 2 + 4x + 12)(x − 3) + 29 [illegible] equating coefficients. Set x = 3 to find the value of D. Set x = 0 and use your value of D to find the value of C. You can find the remaining values by equating coefficients of x3 and x2. [illegible] 12 + ______ x−3 Example 13 x 4 + x 3 + x − 10 f(x) = ___________ x 2 + 2x − 3 Dx + E Show that f(x) can be written as Ax 2 + Bx + C + ________ and find the values of A, B, C, D and E [illegible] 0x2 + x − 10 3x2 3x2 + x 2x2 + 3x 5x2 − 2x − 10 5x2 + 10x − 15 −12x + 5 −12x + 5 ≡ x 2 − x + 5 + ___________ x 2 + 2x − 3 Watch out When you are dividing by a quadratic expres[illegible] remainder is −12x + 5. Write the remainder as a fraction over the whole divisor. So A = 1, B = −1, C = 5, D = −12 and E = 5. 1F Exercise E 1 E 2 E 3 x + 2x + 3x − 4 ____________ [illegible] values of a, b, c and d. x+3 x+3 x3 − 8 f(x) = ______ x−2 Show that f(x) can be written in the form px 2 + qx + r and find the values of p, q and r. E 4 nx + p 2x 2 + 4x + 5 ≡ m + ______ [illegible] (2x 2 + 2) + Cx + D E E E 6 7 8 16 4x − 5x + 3x − 14 ________________ 3 2 x2 + 2x − 1 (4 marks) (4 marks) (4 marks) (4 marks) (4 marks) Cx + D ≡ Ax + B + _________ x 2 + 2x − 1 Fi[illegible] x2 + 1 and find the values of p, q, r, s and t. (4 marks) 2x 4 + 3x 3 − 2x 2 + 4x − 6 dx + e ≡ ax 2 + bx + c + _________ find the values Given that ____________________ x2 + x − 2 x2 + [illegible] Bx + C)(x2 − 3) + Dx + E E/P 10 a Fully factorise the expression x 4 − 1. −1 b Hence, or otherwise, write the algebraic fraction ______ in the form x+1 (ax + b)(cx2 + dx + e) and find [illegible] numerator by the denominator. Remember an improper algebraic fraction is one where the degree of the numerator is greater than or equal to the degree of the denominator. Example [illegible] 3x − 2 ≡ ____________ (x − 1)(x − 2) x2 − 3x + 2 3x2 − 3x − 2 ____________ 3 ≡ x2 − 3x + 2 3x2 − 3x − 2 3x2 − 9x + 6 6x − 8 Divide the denominator into the numerator. It goes in 3 [illegible] 8 C B Let ___________ ≡ ______ + ______ (x − 1)(x − 2) (x − 1) (x − 2) B(x − 2) + C(x − 1) ≡ _________________ (x − 1)(x − 2) 6x − 8 ≡ B(x − 2) + C (x − 1) Let x = 2: 12 − 8 = B × 0 + [illegible] and C = 4. + 4 _______ 3x2 − 3x − 2 Write ____________ as a mixed fraction. (x − 1)(x − 2) Factorise x2 − 3x + 2. The denominators must be (x − 1) and (x − 2). Add the two fractions. T[illegible] Chapter 1 1G Exercise E E E 1 2 3 C x 2 + 3x − 2 B g(x) = ___________ . Show that g(x) can we written in the form A + _____ + _____ x−1 x−2 (x + 1)(x − 3) and find the values of the c[illegible] Find the values of the constants A, B, C and D in the following identity: x −x −x−3 _____________ 3 2 x(x − 1) E 4 E 5 E P 6 7 C D ≡ Ax + B + __ + _____ x x−1 C −3x3 − 4x2 + 19x + 8 D[illegible] marks) 4x 2 + 25 p(x) ≡ ________ 4x 2 − 25 C B Show that p(x) can be written in the form A + ______ + ______, where A, B and C 2x − 5 2x + 5 are constants to be found. (4 marks) C 2x[illegible] express the following as partial fractions: 4x2 + 17x − 11 a ____________ x2 + 3x − 4 E 8 E 9 x4 − 4x3 + 9x2 − 17x + 12 b ____________________ x3 − 4x2 + 4x C 6x 3 − 7x 2 + 3 D ≡ [illegible] 4x + 1 C D Show that q(x) can we written in the form Ax + B + ______ + ________2 and find the 2x − 1 (2x − 1) values of the constants A, B, C and D. E (5 marks) x 4 + 2x 2 − 3x + 8 10[illegible]

[illegible]

×(−2) ×(−2) Here the common ratio is −2. The sequence alternates between positive and negative terms. Notation A geometric sequence with a common ratio |r| < 1 converges. This means it tends to a certain value. You call the value the limit of the sequence. Notation ■ The formula for the nth term of a geometric sequence is: un = arn − 1 where a is the first term and r is the common ratio. 66 An alternating sequence is a sequence in which terms are alternately positive and negative. Sequences and series Example 8 Find the i 10th and ii nth terms in the following geometric sequences: a 3, 6, 12, 24, ... b 40, −20, 10, −5, ... For this sequence a = 3 and r = _63 = 2. a 3, 6, 12, 24, ... i 10th term = 3 × 29 = 3 × 512 = 1536 ii nth term = 3 × 2n−1 For the 10th term use arn−1 with a = 3, r = 2 and n = 10. For the nth term use arn−1 with a = 3 and r = 2. 20 _1 For this sequence a = 40 and r = − __ 40 = − 2 b 40, −20, 10, −5, ... 1 i 10th term = 40 × (− __ 2) 9 Use arn−1 with a = 40, r = − _12 and n = 10. ___ 1 = 40 × − 512 5 = − ___ 64 1 ii nth term = 40 × (− __ 2) n−1 1 = 5 × 8 × (− __ 2) =5× Example × (− 2) 5 × ____ 2n−4 23 = (−1)n−1 n−1 __1 n−1 Use arn−1 with a = 40, r = − _12 and n = n. xm 1 Use laws of indices __n = x(m − n) = ____ x xn−m 1 1 So 23 × ____ = _____ 2n−1 2n−1−3 9 The 2nd term of a geometric sequence is 4 and the 4th term is 8. Given that the common ratio is positive, find the exact value of the 11th term in the sequence. nth term = arn−1, so the 2nd term is ar, and the 4th term is ar3 ar = 4 ar3 = 8 (1) (2) Dividing equation (2) by equation (1): Problem-solving You can use the general term of a geometric sequence to write two equations. Solve these simultaneously to find a and r, then find the 11th term in the sequence. 8 = __ 4 ar ar3 ___ r2 = 2 __ r = √2 You are told in the question that r > 0 so use the positive square root. 67 Chapter 3 Substituting back into equation (1): __ a√ 2 = 4 4__ a = ___ √2 __ a = 2√ 2 Rationalise the denominator. nth term = arn − 1, so ____ 11th term = (2√ 2)(√ 2)10 __ = 64√ 2 Simplify your answer as much as possible. Example 10 The numbers 3, x and (x + 6) form the first three terms of a geometric sequence with all positive terms. Find: a the possible values of x, b the 10th term of the sequence. u3 ___ u1 = u2 x+6 x ______ = 3 x 2 x = 3(x + 6) x2 = 3x + 18 x2 − 3x − 18 = 0 (x − 6)(x + 3) = 0 x = 6 or −3 So x is either 6 or −3, but there are no negative terms so x = 6. u2 ___ a b 10th term = ar9 = 3 × 29 = 3 × 512 = 1536 The 10th term is 1536. Problem-solving In a geometric sequence the ratio between u3 u2 __ consecutive terms is the same, so __ u1 = u 2 Simplify the algebraic fraction to form a quadratic equation. ← Year 1, Section 3.2 Factorise. If there are no negative terms then −3 cannot be an answer. Use the formula nth term = arn−1 with n = 10, x 6 a = 3 and r = __ = __ = 2. 3 3 Example 11 What is the first term in the geometric progression 3, 6, 12, 24, ... to exceed 1 million? nth term = arn−1 = 3 × 2n−1 We want nth term > 1 000 000 Problem-solving Determine a and r, then write an inequality using the formula for the general term of a geometric sequence. Sequence has a = 3 and r = 2. 68 Sequences and series 3 × 2n−1 > 1 000 000 So 2n−1 log 2n−1 > Divide by 3. 1 000 000 _________ > log(3) 1 000 000 _________ (n − 1) log2 > log(3) 1 000 000 _________ 3 1 000 000 log(_________) 3 n − 1 > _____________ log(2) n − 1 > 18.35 (2 d.p.) n > 19.35 n = 20 The 20th term is the first to exceed 1 000 000. Exercise To solve this inequality take logs of both sides. log an = n log a ← Year 1, Chapter 14 Divide by log 2. n has to be an integer. Online Use your calculator to check your answer. 3C 1 Which of the following are geometric sequences? For the ones that are, give the value of the common ratio, r. a 1, 2, 4, 8, 16, 32, ... b 2, 5, 8, 11, 14, ... c 40, 36, 32, 28, ... d 2, 6, 18, 54, 162, ... e 10, 5, 2.5, 1.25, ... f 5, −5, 5, −5, 5, ... g 3, 3, 3, 3, 3, 3, 3, ... h 4, −1, 0.25, −0.0625, ... 2 Continue the following geometric sequences for three more terms. a 5, 15, 45, ... b 4, −8, 16, ... 1 1 c 60, 30, 15, ... d 1, _4 , __ 16 , ... 2 f x, −2x2, 4x3, ... e 1, p, p , ... P 3 If 3, x and 9 are the first three terms of a geometric sequence, find: a the exact value of x, b the exact value of the 4th term. Problem-solving In a geometric sequence the common u3 u 2 __ ratio can be calculated by __ u1 or u 2 4 Find the sixth and nth terms of the following geometric sequences. a 2, 6, 18, 54, ... b 100, 50, 25, 12.5, ... c 1, −2, 4, −8, ... d 1, 1.1, 1.21, 1.331, ... 5 The nth term of a geometric sequence is 2 × 5n. Find the first and 5th terms. 6 The sixth term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio. 69 Chapter 3 7 A geometric sequence has first term 4 and third term 1. Find the two possible values of the 6th term. E/P 8 The first three terms of a geometric sequence are given by 8 − x, 2x, and x2 respectively where x > 0. a Show that x3 − 4x2 = 0. (2 marks) b Find the value of the 20th term. (3 marks) c State, with a reason, whether 4096 is a term in the sequence. E/P (1 mark) 9 A geometric sequence has first term 200 and a common ratio p where p > 0. The 6th term of the sequence is 40. a Show that p satisfies the equation 5 log p + log 5 = 0. (3 marks) b Hence or otherwise, find the value of p correct to 3 significant figures. (1 mark) P 10 A geometric sequence has first term 4 and fourth term 108. Find the smallest value of k for which the kth term in this sequence exceeds 500 000. P 11 The first three terms of a geometric sequence are 9, 36, 144. State, with a reason, whether 383 616 is a term in the sequence. Problem-solving Determine the values of a and r and find the general term of the sequence. Set the number given equal to the general term and solve to find n. If n is an integer, then the number is in the sequence. P 12 The first three terms of a geometric sequence are 3, −12, 48. State, with a reason, whether 49 152 is a term in the sequence. P 13 Find which term in the geometric progression 3, 12, 48, ...is the first to exceed 1 000 000. 3.4 Geometric series A geometric series is the sum of the terms of a geometric sequence. 3, 6, 12, 24, ... is a geometric sequence. 3 + 6 + 12 + 24 + ... is a geometric series. ■ The sum of the first n terms of a geometric series is given by the formula a(1 − r n) Sn = _______, r ≠ 1 1−r a(r n − 1) or Sn = _______ , r ≠ 1 r−1 where a is the first term and r is the common ratio. 70 Hint These two formulae are equivalent. It is often easier to use the first one if r < 1 and the second one if r > 1. Sequences and series Example 12 A geometric series has first term a and common difference r. Prove that the sum of the first n terms a(1 − rn) _______ of this series is given by Sn = 1−r Sn = a + ar + ar2 + ar3 + ... arn−2 + arn−1 (1) Let rSn = ar + ar2 + ar3 + ... arn−1 + arn (2) (1) − (2) gives Sn − rSn = a − arn Sn(1 − r) = a(1 − rn) a(1 − rn) Sn = _______ 1−r Multiply by r. Subtract rSn from Sn . Take out the common factor. Divide by (1 − r). Problem-solving You need to learn this proof for your exam. Example 13 Find the sums of the following geometric series. a 2 + 6 + 18 + 54 + ... (for 10 terms) b 1024 − 512 + 256 − 128 + ... + 1 a Series is 2 + 6 + 18 + 54 + ... (for 10 terms) So a = 2, r = 6 __ 2 As in all questions, write down what is given. = 3 and n = 10 2(310 − 1) So S10 = ________ = 59 048 3−1 b Series is 1024 − 512 + 256 − 128 + ... + 1 When r > 1 it is easier to use the formula a(rn − 1) Sn = ______ r−1 512 __1 So a = 1024, r = − ____ 1024 = − 2 and the nth term = 1 1 1024 (− __ 2) n−1 =1 First solve arn−1 = 1 to find n. (−2)n−1 = 1024 2n−1 = 1024 log 1024 n − 1 = ________ log 2 n − 1 = 10 n = 11 So (−2)n−1 = (−1)n−1(2n−1) = 1024, so (−1)n−1 must be positive and 2n−1 = 1024. 1024 = 210 1 11 1024(1 − (−__ 2)) ______________ Sn = __1 1 − (− 2) 1 1024(1 + ____ 2048) _____________ = __1 1+ When r < 1, it is easier to use the formula a(1 − rn) Sn = ______ 1−r 2 1024.5 = ______ = 683 __ 3 2 71 Chapter 3 Example 14 Find the least value of n such that the sum of 1 + 2 + 4 + 8 + ... to n terms exceeds 2 000 000. 1 (2n − 1) Sum to n terms is Sn = _______ 2−1 n =2 −1 If this is to exceed 2 000 000 then Sn > −1> 2n > n log 2 > 2n 2 000 000 2 000 000 2 000 001 log(2 000 001) log(2 000 001) n > _____________ log(2) n > 20.9 It needs 21 terms to exceed 2 000 000. Exercise Problem-solving Determine the values of a and r, then use the formula for the sum of a geometric series to form an inequality. Add 1. Use laws of logs: log an = n log a. Round n up to the nearest integer. 3D 1 Find the sum of the following geometric series (to 3 d.p. if necessary). a 1 + 2 + 4 + 8 + ... (8 terms) 8 256 2 4 c

r. The sum of the first three terms of the series is 7. ← Sections 3.6, 3.7 (2) The company has a charity scheme whereby they match any employee charity contribution exactly. d Given that the average employee charity contribution is £5 each year, find the total charity donation over the 10-year period from the end of year 1 to the end of year 10. Give your answer to the nearest £1000. (3) ← Section 3.8 Review exercise 1 E/p 30 A geometric series is given by E 6 − 24x + 96x − …. 2 The series is convergent. a Write down a condition on x. ∞ Given that ∑6 × (− 4x) r=1 r−1 (1) = __ 5 24 ← Sections 3.5, 3.6 (5) ← Section 4.2 E 1 ___ 31 g(x) = _____ √1 − x a Show that the series expansion of g(x) up to and including the x3 term is x 3 x2 5 x3 1 + __ + ___ + ___ (5) 2 8 16 b State the range of values of x for which the expansion is valid. (1) ____ 35 h(x) = √ 4 − 9x , |x| < _9 4 32 When (1 + ax)n is expanded as a series in ascending powers of x, the coefficients of x and x2 are −6 and 45 respectively. a Find the series expansion of h(x), in ascending powers of x, up to and including the x2 term. Simplify each term. (4) 1 b Show that, when x = ____, the exact ___ 100 √ 391 ____ (2) value of h(x) is 10 c Use the series expansion in part a 1 to estimate the value of h(___) and 100 state the degree of accuracy of your approximation. (3) a Find the value of a and the value of n. ← Section 4.2 ← Section 4.1 p 3 (5) b Calculate the value of x. E 34 f(x) = (1 + x) (3 + 2x) −3, |x| < _2 Find the binomial expansion of f(x) in ascending powers of x, up to and including the term in x3. Give each coefficient as a simplified fraction. b Find the coefficient of x . 3 c Find the set of values of x for which the expansion is valid. E/p expansion _4 + _4 x + cx 2 + ... 1 ← Section 4.1 E ____ value of (1 + 4x) 2 is 112 √ 112 ______ 1000 (2) 3 c Substitute x = ____ into the binomial 100 expansion in part a and hence obtain ___ an approximation to √ 112 . Give your answer to 5 decimal places. (3) 1 a Find the values of the constants a, b and c. 33 a Find the binomial expansion of _3 (1 + 4x) 2 in ascending powers of x up to and including the x3 term, simplifying each term. (4) 3 b Show that, when x = ____, the exact 100 _3 36 Given that (a + bx) −2 has binomial (4) b Find the coefficient of the x3 term in the expansion. (2) ← Section 4.2 E/p 3 + 5x 1 37 g(x) = __________ , |x| < __3 (1 + 3x)(1 − x) Given that g(x) can be expressed in the A B form g(x) = ______ + _____ 1 + 3x 1 − x a find the values of A and B. (3) d Calculate the percentage error in your estimate to 5 decimal places. (2) b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the x2 term. Simplify each term. (6) ← Section 4.1 ← Sections 4.1, 4.3 111 Review exercise 1 E/p 3x − 1 A 1 B 38 _______2 ; _____ + _______2 , |x| < __ 2 1 − 2x (1 − 2x) (1 − 2x) a Find the values of A and B. (3) 3x − 1 b Hence, or otherwise, expand _______2 (1 − 2x) in ascending powers of x, as far as the term in x3. Give each coefficient as a simplified fraction. (6) ← Sections 4.1, 4.3 E/p 25 39 f(x) = ____________ , |x| < 1 (3 + 2x) 2 (1 − x) Challenge 1 The functions f and g are defined by f(x) = −3|x + 3| + 15, x ∈ ℝ g(x) = − _34 x + _32 , x ∈ ℝ The diagram shows a sketch of the graphs y = f(x) and y = g(x), which intersect at points A and B. M is the midpoint of AB. The circle C, with centre M, passes through points A and B, and meets y = f(x) at point P as shown in the diagram. y = f(x) f(x) can be expressed in the form C B A _____ + ______ + _____ 3 + 2x (3 + 2x)2 1 − x a Find the values of A, B and C. A (4) b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in (6) x2. Simplify each term. ← Sections 4.1, 4.2, 4.3 E/p y C 4 x 2 + 30x + 31 B 40 ____________ = A + ____ + _____ x + 4 2x + 3 (x + 4)(2x + 3) a Find the values of the constants A, B and C. (4) b Hence, or otherwise, expand 4 x 2 + 31x + 30 ___________ in ascending powers of (x + 4) (2x + 3) x, as far as the term in x2. Give each coefficient as a simplified fraction. (7) C P O B x y = g(x) a Find the equation of the circle. b Find the area of the triangle APB. ← Section 2.6 11 15 i=6 i=12 2 Given that an+1 = an + k, a1 = n and ∑ai = ∑ai, show that n = _52 k. ← Section 3.6 3 The diagram shows a sketch of the functions p(x) = |x2 − 8x + 12| and q(x) = |x2 − 11x + 28|. y y = p(x) y = q(x) A ← Sections 4.1, 4.2, 4.3 B C O x Find the exact values of the x-coordinates of the points A, B and C. ← Section 2.5 112 5 Radians Objectives After completing this unit you should be able to: ● Convert between degrees and radians and apply this to trigonometric graphs and their transformations → pages 114–116 ● Know exact values of angles measured in radians → pages 117–118 ● Find an arc length using radians → pages 118–122 ● Find areas of sectors and segments using → pages 122–128 radians ● Solve trigonometric equations in radians → pages 128–132 ● Use approximate trigonometric values when θ is small → pages 133–135 Prior knowledge check 1 Write down the exact values of the following trigonometric ratios. a cos 120° b sin 225° c tan (−300°) d sin (−480°) 2 ← Year 1, Chapter 10 Simplify each of the following expressions. a (tan θ cos θ)2 + cos2 θ 1 b 1 − ____ c cos2θ 3 __________ sin θ cos θ √1 − _______ tan θ ← Year 1, Chapter 10 Show that a (sin 2θ + cos 2θ)2 ≡ 1 + 2 sin 2θ cos 2θ 2 cos2θ 2 b ___ − 2 sin θ ≡ _____ sin θ sin θ ← Year 1, Chapter 10 4 Radians are units for measuring angles. They are used in mechanics to describe circular motion, and can be used to work out the distances between the pods around the → Exercise 5B Q13 edge of a Ferris wheel. Solve the following equations for θ in the interval 0 < θ < 360°, giving your answers to 3 significant figures where they are not exact. a 4 cos θ + 2 = 3 c 6 tan2 θ b 2 sin 2θ = 1 + 10 tan θ − 4 = tan θ d 10 + 5 cos θ = 12 sin2 θ ← Year 1, Chapter 10 113 Chapter 5 5.1 Radian measure 1 So far you have probably only measured angles in degrees, with one degree representing ___ of a 360 complete revolution or circle. You can also measure angles in units called radians. 1 radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. If the arc AB has length r, then ∠AOB is 1 radian. Links A r O 1 rad r r B You always use radian measure when you are differentiating or integrating trigonometric functions. → Sections 9.1, 11.1 Notation The circumference of a circle of radius r is an arc of length 2π r, so it subtends an angle of 2π radians at the centre of the circle. ■ 2π radians = 360° ■ π radians = 180° You can write 1 radian as 1 rad. 2πr r 2π rad Hint This means that 1 radian = 57.295...° 180° ■ 1 radian = _____ π Example 1 Convert the following angles into degrees. 7π 4π a ___ rad b ___ rad 8 15 7π a ___ rad 8 7 = __ × 180° 8 = 157.5° Example 4π b ___ rad 15 180° = 4 × _____ 15 = 48° 180° 180° ______ 1 radian = _____ π , so multiply by π 180° __ 7 7π ___ × ____ π = 8 × 180° 8 2 Convert the following angles into radians. Leave your answers in terms of π. a 150° π a 150° = 150 × ____ rad 180 5π = ___ rad 6 π b 110° = 110 × ____ rad 180 11π = ___ rad 18 114 b 110° π π 1° = ____ radians, so multiply by ____ 180 180 Your calculator will often give you exact answers in terms of π. Radians You should learn these important angles in radians: π ■ 30° = __ radians π ■ 60° = __ radians 3 π ■ 90° = __ radians 2 6 π ■ 45° = __ radians 4 Example ■ 180° = π radians ■ 360° = 2π radians Online 3 Find: a sin (0.3 rad) b cos (π rad) c tan (2 rad) Give your answers correct to 2 decimal places where appropriate. a sin (0.3 rad) = 0.30 (2 d.p.) b cos (π rad) = −1 Use your calculator to evaluate trigonometric functions in radians. Watch out You need to make sure your calculator is in radians mode. c tan (2 rad) = −2.19 (2 d.p.) Example 4 Sketch the graph of y = sin x for 0 < x < 2π. If the range includes values given in terms of π, you can assume that the angle has been given in radians. y 1 π sin (__) = sin 90° = 1 2 y = sin x 0.5 O - 0.5 π 2 π 3π 2 2π x -1 Example 5 Sketch the graph of y = cos (x + π) for 0 < x < 2π. y 1 y = cos (x + π) The graph of y = cos (x + a) is a translation of the −a graph y = cos x by the vector ( ). 0 0.5 O - 0.5 -1 π 3 π 3π 2 2π x y = cos x 115 Chapter 5 Exercise 5A 1 Convert the following angles in radians to degrees. 5π 5π π a ___ c ___ d ___ b ___ 20 4 15 12 3π e ___ 2 f 3π 2 Convert the following angles to degrees, giving your answer to 1 d.p. a 0.46 rad b 1 rad __ d √ 3 rad c 1.135 rad 3 Evaluate the following, giving your answers to 3 significant figures. __ b cos (√ 2 rad) a sin (0.5 rad) c tan (1.05 rad) d sin (2 rad) e sin (3.6 rad) 4 Convert the following angles to radians, giving your answers as multiples of π. a 8° b 10° c 22.5° d 30° f 240° g 270° h 315° i 330° e 112.5° 5 Convert the following angles to radians, giving your answers to 3 significant figures. a 50° b 75° c 100° d 160° e 230° f 320° 6 Sketch the graphs of: a y = tan x for 0 < x < 2π b y = cos x for −π < x < π Mark any points where the graphs cut the coordinate axes. 7 Sketch the following graphs for the given ranges, marking any points where the graphs cut the coordinate axes. a y = sin (x − π) for −π < x < π π c y = tan (x + __) for −π < x < π 2 E/P b y = cos 2x for 0 < x < 2π 1 d y = sin __3 x + 1 for 0 < x < 6π 2π 8 The diagram shows the curve with equation y = cos (x − ___), −2π < x < 2π. 3 y Problem-solving O x Make sure you write down the coordinates of all four points of intersection with the x-axis and the coordinates of the y-intercept. Write down the coordinates of the points at which the curve meets the coordinate axes. (3 marks) Challenge Describe all the angles, θ, in radians, that satisfy: a cos θ = 1 b sin θ = −1 c tan θ is undefined. 116 Hint You can use nπ, where n is an integer, to describe any integer multiple of π. Radians You need to learn the exact values of the trigonometric ratios of these angles measured in radians: __ __ √3 √3 π __ π ___ π ___ 1 1____ ___ __ ■ sin = ■ cos = ■ tan = = 2 3 6 2 6 6 √3 __ ■ π √3 sin__ = ___ 3 π __ 1 ■ cos__ = 2 3 __ 2 π ___ 1 ■ sin__ = __ = __ 4 √ __ 2 3 __ 2 π ___ 1 ■ cos__ = __ = ___ 4 2 √2 π √ ■ tan__ = 3 √2 √ π ■ tan__ =1 4 2 You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ. ■ sin (π − θ) = sin θ ■ sin (π + θ) = −sin θ ■ sin (2π − θ) = −sin θ ■ cos (π − θ) = −cos θ ■ cos (π + θ) = −cos θ ■ cos (2π − θ) = cos θ ■ tan (π − θ) = −tan θ ■ tan (π + θ) = tan θ ■ tan (2π − θ) = −tan θ Example y S π−θ θ θ π+θ A θ θ θ O T x 2π - θ C Links The CAST diagram shows you which trigonometric ratios are positive in which quadrant. You can also use the symmetry properties and periods of the graphs of sin, cos and tan to find these results. ← Year 1, Chapter 10 6 Problem-solving Find the exact values of: −7π 4π a cos ___ b sin(___) 3 6 You can also use the symmetry properties of y = cos x: y y = cos x 1 π a 2 4π 3 4π 3 2π 3 π 0 3π 2 4π π π cos ___ = cos (π + __) = −cos __ 3 3 3 4π 1 So cos ___ = − __ 2 3 O π π 3 2 π 3π 2 2π x -1 4π __ π ___ is bigger than π. 3 3 Use cos (π + θ) = −cos θ. 117 Chapter 5 b π 2 5π 6 π 0 7π -6 3π 2 7π 5π sin (− ___) = sin __ 6 6 7π 1 So sin (− ___) = − __ 6 2 Exercise π = sin __ 6 Use sin (π − θ) = sin θ. 5B π π π 1 Express the following as trigonometric ratios of either __, __ or __, and hence find their exact values. 3 6 4 3π π 11π ____ c sin ____ b sin (−) a sin 4 3 6 5π 5π 2π d cos ___ e cos ___ f cos ___ 3 3 4 3π 5π 7π g tan ___ i tan ___ h tan (− ___) 4 4 6 2 Without using a calculator, find the exact values of the following trigonometric ratios. 5π 7π 7π c cos (− ___) a sin ___ b sin (− ___) 3 3 6 5π 11π 2π d cos ____ e tan ____ f tan (− ___) 4 3 3 P C 3 The diagram shows a right-angled triangle ACD on another __ 2√ 6 ____ right-angled triangle ABC with AD = and BC = 2. 3 D __ Show that DC = k√ 2 , where k is a constant to be determined. 2 6 3 2 A π 3 B 5.2 Arc length Using radians greatly simplifies the formula for arc length. ■ To find the arc length l of a circle use the formula l = rθ, where r is the radius of the circle and θ is the angle, in radians, contained by the sector. 118 r θ l Radians Example 7 Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 rad at the centre of the circle. Arc length = 5.2 × 0.8 = 4.16 cm Example Online Explore the arc length of a sector using GeoGebra. Use l = r θ, with r = 5.2 and θ = 0.8. 8 An arc AB of a circle with radius 7 cm and centre O has a length of 2.45 cm. Find the angle ∠AOB subtended by the arc at the centre of the circle. A 7cm O θ 2.45cm 7cm B l= 2.45 = 2.45 _____ = 7 θ= Example rθ 7θ Use l = rθ, with l = 2.45 and r = 7. θ 0.35 rad Using this formula gives the angle in radians. 9 An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ. r cm O A θ r cm B P = rθ + 2r = r(2 + θ) P So r = _______ (2 + θ) Problem-solving When given a problem in words, it is often a good idea to sketch and label a diagram to help you to visualise the information you have and what you need to find. The perimeter = arc AB + OA + OB, where arc AB = rθ. Factorise. 119 Chapter 5 Example 10 C The border of a garden pond consists of a straight edge AB of length 2.4 m, and a curved part C, as shown in the diagram. The curved part is an arc of a circle, centre O and radius 2 m. Find the length of C. θ 2m A O 2 A 2.4 m 2m B Online Explore the area of a sector using GeoGebra. x Problem-solving B 1.2 O Look for opportunities to use the basic trigonometric ratios rather than the more complicated cosine rule or sine rule. AOB is an isosceles triangle, so you can divide it into congruent right-angled triangles. Make sure your calculator is in radians mode. 1.2 sin x = ___ 2 x = 0.6435... rad Acute ∠AOB = 2x rad = 2 × 0.6435... = 1.2870... rad So θ = (2π − 1.2870...) rad = 4.9961... rad So length of C = 9.99 m (3 s.f.) C subtends the reflex angle θ at O, so length of C = 2θ. θ + acute ∠AOB = 2π rad Exercise C = 2θ 5C 1 An arc AB of a circle, centre O and radius r cm, subtends an angle θ radians at O. The length of AB is l cm. P a Find l when: i r = 6, θ = 0.45 ii r = 4.5, θ = 0.45 b Find r when: i l = 10, θ = 0.6 ii l = 1.26, θ = 0.7 c Find θ when: i l = 10, r = 7.5 ii l = 4.5, r = 5.625 2 A minor arc AB of a circle, centre O and radius 10 cm, subtends an angle x at O. The major arc AB subtends an angle 5x at O. Find, in terms of π, the length of the minor arc AB. 3 iii r = 20, θ = __ π 8 5 iii l = 1.5π, θ = ___ π 12 ___ iii l = √ 12 , r = √ 3 Notation The minor arc AB is the shorter arc between points A and B on a circle. 3 An arc AB of a circle, centre O and radius 6 cm, has length l cm. Given that the chord AB has length 6 cm, find the value of l, giving your answer in terms of π. ___ __ 4 The sector of a circle of radius √ 10 cm contains an angle of √ 5 radians, as shown in the diagram. Find the length of the arc, giving your answer __ in the form p√ q cm, where p and q are integers. 120 10 cm 5 rad 10 cm Radians P 5 2 cm Referring to the diagram, find: a the perimeter of the shaded region when θ = 0.8 radians. 3 cm O b the value of θ when the perimeter of the shaded region is 14 cm. Problem-solving θ The radius of the larger arc is 3 + 2 = 5 cm. 3 cm 2 cm P 6 A sector of a circle of radius r cm contains an angle of 1.2 radians. Given that the sector has the same perimeter as a square of area 36 cm2, find the value of r. P 7 A sector of a circle of radius 15 cm contains an angle of θ radians. Given that the perimeter of the sector is 42 cm, find the value of θ. E/P 8 In the diagram AB is the diameter of a circle, centre O and radius 2 cm. The point C is on the circumference such that 2 ∠COB = __ π radians. 3 a State the value, in radians, of ∠COA. C 2π 3 A O B 2 cm (1 mark) The shaded region enclosed by the chord AC, arc CB and AB is the template for a brooch. b Find the exact value of the perimeter of the brooch. P 9 (5 marks) The points A and B lie on the circumference of a circle with centre O and radius 8.5 cm. The point C lies on the major arc AB. Given that ∠ACB = 0.4 radians, calculate the length of the minor arc AB. E/P a Write down, in terms of R and r, the length of OC. (1 mark) b Using △OCE, show that R sin θ = r (1 + sin θ). (3 marks) 3 c Given that sin θ = __ and that the perimeter of the sector OAB is 4 21 cm, find r, giving your answer to 3 significant figures. (7 marks) P O 10 In the diagram OAB is a sector of a circle, centre O and radius R cm, and ∠AOB = 2θ radians. A circle, centre C and radius r cm, touches the arc AB at T, and touches OA and OB at D and E respectively, as shown. 11 The diagram shows a sector AOB. The perimeter of the sector is twice the length of the arc AB. Find the size of angle AOB. R cm r cm D E C A B A T O B P 12 A circular Ferris wheel has 24 pods equally spaced on its circumference. 3π Given the arc length between each pod is ___ m, and modelling each pod as a particle, 2 a calculate the diameter of the Ferris wheel. Given that it takes approximately 30 seconds for a pod to complete one revolution, b estimate the speed of the pod in km/h. 121 Chapter 5 E/P 13 The diagram above shows a triangular garden, PQR, with PQ = 12 m, PR = 7 m and ∠QPR = 0.5 radians. The curve SR is a small path separating the shaded patio area and the lawn, and is an arc of a circle with centre at P and radius 7 m. P 12 m Find: a the length of the path SR (2 marks) 7m S b the perimeter of the shaded patio, giving your answer to 3 significant figures. (4 marks) E/P 0.5 rad R Q 14 The shape XYZ shown is a design for an earring. X 15 mm 5 mm Y 5 mm O 1.1 rad Z The straight lines XY and XZ are equal in length. The curve YZ is an arc of a circle with centre O and radius 5 mm. The size of ∠YOZ is 1.1 radians and XO = 15 mm. a Find the size of ∠XOZ, in radians, to 3 significant figures. (2 marks) b Find the total perimeter of the earring, to the nearest mm. (6 marks) 5.3 Areas of sectors and segments Using radians also greatly simplifies the formula for the area of a sector. ■ To find the area A of a sector of a 1 circle use the formula A = __r2 θ, 2 where r is the radius of the circle and θ is the angle, in radians, contained by the sector. Notation r θ A A sector of a circle is the portion of a circle enclosed by two radii and an arc. The smaller area is known as the minor sector and the larger is known as the major sector. Example 11 Find the area of the sector of a circle of radius 2.44 cm, given that the sector subtends an angle of 1.4 radians at the centre of the circle. 1 Area of sector = __ × 2.442 × 1.4 2 = 4.17 cm2 (3 s.f.) 122 1 Use A = __ r2 θ with r = 2.44 and θ = 1.4. 2 Radians Example 12 In the diagram, the area of the minor sector AOB is 28.9 cm2. Given that ∠AOB = 0.8 radians, calculate the value of r. 1 28.9 = __ r2 × 0.8 = 0.4r2 2 28.9 So r2 = _____ = 72.25 0.4 ______ r = √ 72.25 = 8.5 r cm O A 0.8 r cm B 1 Let area of sector be A cm2, and use A = __ r2 θ. 2 Use the positive square root in this case as a length cannot be negative. Example 13 A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of the plot of land. A Online Explore the area of a segment using GeoGebra. 55 m θ 55 m B Arc AB = 176 − (55 + 55) = 66 m 66 = 55θ So θ = 1.2 radians 1 Area of plot = __ × 552 × 1.2 2 = 1815 m2 O Draw a diagram including all the data and let the angle of the sector be θ. Problem-solving In order to find the area of the sector, you need to know θ. Use the information about the perimeter to find the arc length AB. As the perimeter is given, first find length of arc AB. Use the formula for arc length, l = rθ. 1 Use the formula for area of a sector, A = __ r2 θ. 2 You can find the area of a segment by subtracting the area of triangle OPQ from the area of sector OPQ. P r cm O θ Q A 1 1 Using __r2θ for the area of the sector and __ab sin θ for the area of a triangle: 2 2 1 1 A = __r2θ − __r2 sin θ 2 2 1 = __r2(θ − sin θ) 2 1 2 ■ The area of a segment in a circle of radius r is A = __ r (θ − sin θ) 2 123 Chapter 5 Example 14 The diagram shows a sector of a circle. Find the area of the shaded segment. 7 cm 1.2 rad 7 cm Area of segment = 1 __ × 72(1.2 − sin 1.2) 2 1 = __ × 49 × 0.26796... 2 = 6.57 cm2 (3 s.f.) Use A = _12 r2(θ − sin θ) with r = 7 and θ = 1.2 radians. Make sure your calculator is in radians mode when calculating sin θ. Example 15 O 4m B 4m 5m A In the diagram above, OAB is a sector of a circle, radius 4 m. The chord AB is 5 m long. Find the area of the shaded segment. Calculate angle AOB first: 42 + 42 − 52 cos∠AOB = _____________ 2×4×4 7 = ___ 32 So ∠AOB = 1.3502... Area of shaded segment 1 = __ × 42(1.3502... − sin 1.3502...) 2 1 = __ × 16 × 0.37448... 2 = 3.00 m2 (3 s.f.) 124 Problem-solving In order to find the area of the segment you need to know angle AOB. You can use the cosine rule in triangle AOB, or divide the triangle into two right-angled triangles and use the trigonometric ratios. Use the cosine rule for a non-right-angled triangle. Watch out Use unrounded values in your calculations wherever possible to avoid rounding

[illegible]

and that angle A is obtuse, find the exact values of: 5 a sec A b sin A a Using 1 + tan2 A ≡ sec2 A 25 169 ___ sec2 A = 1 + ___ 144 = 144 13 sec A = 6 ___ 12 25 tan2 A = ___ 144 A S Problem-solving T C You are told that A is obtuse. This means it lies in the second quadrant, so cos A is negative, and sec A is also negative. 13 sec A = − __ 12 sin A b Using tan A ≡ _____ cos A sin A ≡ tan A cos A So 5 __ 12 sin A = (− __ 12) × (− 13) 5 = ___ 13 154 1 12 _____ cos A = − __ 13 , since cos A = sec A Trigonometric functions Example 11 Prove the identities: 1 + cos2 θ a cosec4 θ − cot4 θ ≡ _________ 1 − cos2 θ b sec2 θ − cos2 θ ≡ sin2 θ (1 + sec2 θ) a LHS = cosec4 θ − cot4 θ ≡ (cosec2 θ + cot2 θ) (cosec2 θ − cot2 θ) ≡ cosec2 θ + cot2 θ cos2 θ 1 ≡ _____ + _______ sin2 θ sin2 θ 1 + cos2 θ ≡ ________ sin2 θ ≡ 1 + cos2 θ ________ 1 − cos2 θ ≡ RHS b RHS ≡ sin2 θ + sin2 θ sec2 θ sin2 θ ≡ sin2 θ + ______ cos2 θ ≡ ≡ ≡ ≡ sin2 θ + tan2 θ (1 − cos2 θ) + (sec2 θ − 1) sec2 θ − cos2 θ LHS This is the difference of two squares, so factorise. As 1 + cot2 θ ≡ cosec2 θ, so cosec2 θ − cot2 θ ≡ 1. cos θ 1 Using cosec θ ≡ _____ , cot θ ≡ _____ sin θ sin θ Using sin2 θ + cos2 θ ≡ 1. Write in terms of sin θ and cos θ. 1 Use sec θ ≡ _____. cos θ 2 (cos θ) ≡ tan θ. sin θ sin θ _______ ≡ _____ 2 cos2 θ 2 Look at LHS. It is in terms of cos2 θ and sec2 θ, so use sin2 θ + cos2 θ ≡ 1 and 1 + tan2 θ ≡ sec2 θ. Problem-solving You can start from either the LHS or the RHS when proving an identity. Try starting with the LHS using cos2 θ ≡ 1 − sin2 θ and sec2 θ ≡ 1 + tan2 θ. Example 12 Solve the equation 4 cosec2 θ − 9 = cot θ, in the interval 0 < θ < 360°. The equation can be rewritten as 4(1 + cot2 θ) − 9 = cot θ So 4 cot2 θ − cot θ − 5 = 0 (4 cot θ − 5)(cot θ + 1) = 0 So ∴ This is a quadratic equation. You need to write it in terms of one trigonometrical function only, so use 1 + cot2 θ = cosec2 θ. Factorise, or solve using the quadratic formula. 5 cot θ = __ 4 or cot θ = −1 4 tan θ = __ 5 or tan θ = −1 4 For tan θ = __ 5 155 Chapter 6 S A 38.7° As tan θ is +ve, θ is in the 1st and 3rd quadrants. The acute angle to the horizontal is tan−1 _45 = 38.7°. 38.7° T C If α is the value the calculator gives for tan−1 _45 , then the solutions are α and (180° + α). θ = 38.7°, 210° (3 s.f.) For tan θ = −1 S A tan θ is −ve, θ is in the 2nd and 4th quadrants. The acute angle to the horizontal is tan−1 1 = 45°. A 45° If α is the value the calculator gives for tan−1 (−1), then the solutions are (180° + α) and (360° + α), as α is not in the given interval. 45° C T Online Solve this equation numerically using your calculator. θ = 135°, 315° Exercise 6D Give answers to 3 significant figures where necessary. 1 Simplify each of the following expressions. a 1 + tan2 _2 θ b (sec θ − 1)(sec θ + 1) tan θ sec θ g ________ 1 + tan2 θ h (1 − sin2 θ)(1 + tan2 θ) 1 d (sec2 θ − 1) cot θ e (cosec2 θ − cot2 θ)2 j (sec4 θ − 2 sec2 θ tan2 θ + tan4 θ) P c tan2 θ (cosec2 θ − 1) f 2 − tan2 θ + sec2 θ cosec θ cot θ i __________ 1 + cot2 θ k 4 cosec2 2θ + 4 cosec2 2θ cot2 2θ k 2 Given that cosec x = ______ cosec x , where k > 1, find, in terms of k, possible values of cot x. __ 3 Given that cot θ = −√ 3 , and that 90° < θ < 180°, find the exact values of: a sin θ b cos θ 4 Given that tan θ = _4 , and that 180° < θ < 270°, find the exact values of: 3 a sec θ b cos θ c sin θ 5 Given that cos θ = __ 25 , and that θ is a reflex angle, find the exact values of: 24 a tan θ 156 b cosec θ Trigonometric functions P 6 Prove the following identities. a sec4 θ − tan4 θ ≡ sec2 θ + tan2 θ b cosec2 x − sin2 x ≡ cot2 x + cos2 x c sec2 A(cot2 A − cos2 A) ≡ cot2 A d 1 − cos2 θ ≡ (sec2 θ − 1)(1 − sin2 θ) 1 − tan2 A ≡ 1 − 2 sin2 A e ________ 1 + tan2 A f sec2 θ + cosec2 θ ≡ sec2 θ cosec2 θ g cosec A sec2 A ≡ cosec A + tan A sec A h (sec θ − sin θ)(sec θ + sin θ) ≡ tan2 θ + cos2 θ P 7 Given that 3 tan2 θ + 4 sec2 θ = 5, and that θ is obtuse, find the exact value of sin θ. P 8 Solve the following equations in the given intervals. a sec2 θ = 3 tan θ, 0 < θ < 360° b tan2 θ − 2 sec θ + 1 = 0, −π < θ < π c cosec2 θ + 1 = 3 cot θ, −180° < θ < 180° d cot θ = 1 − cosec2 θ, 0 < θ < 2π e 3 sec _2 θ = 2 tan2 _2 θ, 0 < θ < 360° 1 1 f (sec θ − cos θ)2 = tan θ − sin2 θ, 0 < θ < π __ g tan2 2θ = sec 2θ − 1, 0 < θ < 180° E/P __ h sec2 θ − (1 + √ 3) tan θ + √ 3 = 1, 0 < θ < 2π 9 Given that tan2 k = 2 sec k, a find the value of sec k (4 marks) __ b deduce that cos k = √ 2 − 1. c Hence solve, in the interval 0 < k < 360°, 1 decimal place. E/P (2 marks) tan2 k = 2 sec k, giving your answers to 10 Given that a = 4 sec x, b = cos x and c = cot x, express b in terms of a 16 b show that c 2 = ______ 2 a − 16 E/P E/P (3 marks) 11 Given that x = sec θ + tan θ, 1 a show that ____ x = sec θ − tan θ. 1 b Hence express x2 + __2 + 2 in terms of θ, in its simplest form. x p−1 12 Given that 2 sec2 θ − tan2 θ = p show that cosec2 θ = _____ , p ≠ 2. p−2 (2 marks) (3 marks) (3 marks) (5 marks) (5 marks) 157 Chapter 6 6.5 Inverse trigonometric functions You need to understand and use the inverse trigonometric functions arcsin x, arccos x and arctan x and their graphs. ■ The inverse function of sin x is called arcsin x. y π 2 y = arcsin x Hint 1 x O −1 The sin−1 function on your calculator will give principal values in the same range as arcsin. π −2 The domain of y = arcsin x is −1 < x < 1. π π ● The range of y = arcsin x is − __ < arcsin x < __ or −90° < arcsin x < 90°. 2 2 ● Example 13 Sketch the graph of y = arcsin x. π π y = sin x, − __ < x < __ 2 2 y y = sin x 1 − π π O 2 2 x -1 Step 1 Draw the graph of y = sin x, with the restricted π π domain of − __ < x < __ 2 2 Restricting the domain ensures that the inverse function exists since y = sin x is a one-to-one function for the restricted domain. Only one-to← Section 2.3 one functions have inverses. y = arcsin x y π y = arcsin x 2 O −1 − 158 π 2 1 x Step 2 Reflect in the line y = x. The domain of arcsin x is −1 < x < 1; the range is π π − __ < arcsin x < __ 2 2 Remember that the x and y coordinates of points interchange when reflecting in y = x. For example: π π __ __ (2 , 1) → (1, 2) Trigonometric functions ■ The inverse function of cos x is called arccos x. y π π 2 -1 y = arccos x 1 x O ● The domain of y = arccos x is −1 < x < 1. ● The range of y = arccos x is 0 < arccos x < π or 0° < arccos x < 180°. ■ The inverse function of tan x is called arctan x. y π 2 y = arctan x x O Watch out Unlike arcsin x and arccos x, the function arctan x is defined for all real values of x. -π 2 The domain of y = arctan x is x ∈ ℝ. π π ● The range of y = arctan x is − __ < arctan x < __ or −90° < arctan x < 90°. 2 2 ● Example 14 Work out, in radians, the values of: a arcsin(− __ 2) √2 __ b arccos(−1) a __ c arctan(√ 3) π 2 S A π 4 T - __ π 2 π arcsin(−) = − __ 2 4 √2 __ C π π You need to solve, in the interval − __ < x < __ , 2 2 __ √2 __ the equation sin x = − . 2 π The angle to the horizontal is __ and, as sin is −ve, 4 it is in the 4th quadrant. Online Use your calculator to evaluate inverse trigonometric functions in radians. 159 Chapter 6 b y 1 y = cos x 2 -1 You need to solve, in the interval 0 < x < π, the equation cos x = −1. π x π O Draw the graph of y = cos x. arccos(−1) = π c π 2 S π π You need to solve, in the interval − __ < x < __ , 2 2 __ the equation tan x = √ 3 . π The angle to the horizontal is __ and, as tan is +ve, 3 it is in the 1st quadrant. A π 3 T - π C 2 π arctan(√ 3) = __ 3 You can verify these results using the sin−1, cos−1 and tan−1 functions on your calculator. __ Exercise 6E In this exercise, all angles are given in radians. 1 Without using a calculator, work out, giving your answer in terms of π: a arccos (0) b arcsin(1) c arctan(−1) 1__ e arccos − ___ (√2) 1__ f arctan − ___ (√3) π g arcsin(sin __) 3 2 Find: a arcsin(_2) + arcsin(− _2) 1 P 1 1 1 1 2π h arcsin(sin ___) 3 c arctan(1) − arctan(−1) 3 Without using a calculator, work out the values of: a sin(arcsin _2) b sin(arcsin (− _2)) c tan(arctan(−1)) d cos(arccos 0) 1 P b arccos(_2) − arccos(− _2) d arcsin(− _2) 1 4 Without using a calculator, work out the exact values of: a sin(arccos(2)) _1 __ d sec(arctan(√ 3)) 160 __ b cos(arcsin(− 2)) c tan(arccos(− e cosec(arcsin(−1)) f sin (2arcsin( )) 2 _1 2)) √2 ___ √2 ___ Trigonometric functions P E/P 5 Given that arcsin k = α, where 0 < k < 1 and α is in radians, write down, in terms of α, the first two positive values of x satisfying the equation sin x = k. π 6 Given that x satisfies arcsin x = k, where 0 < k < __ , 2 a state the range of possible values of x (1 mark) b express, in terms of x, i cos k ii tan k π Given, instead, that − __ < k < 0, 2 c how, if at all, are your answers to part b affected? P E/P 7 Sketch the graphs of: π a y = __ + 2 arcsin x 2 c y = arccos (2x + 1) (2 marks) b y = π − arctan x d y = −2 arcsin (−x) 8 The function f is defined as f : x ↦ arcsin x, −1 < x < 1, and the function g is such that g(x) = f(2x). a Sketch the graph of y = f(x) and state the range of f. (3 marks) b Sketch the graph of y = g(x). (2 marks) c Define g in the form g: x ↦ ... and give the domain of g. (3 marks) d Define E/P (4 marks) g−1 in the form g−1: x ↦... _____ 9 a Prove that for 0 < x < 1, arccos x = arcsin √ 1 − x2 b Give a reason why this result is not true for −1 < x < 0. (2 marks) (4 marks) (2 marks) Challenge a Sketch the graph of y = sec x, with the restricted domain π 0 < x < π, x ≠ __ 2 π b Given that arcsec x is the inverse function of sec x, 0 < x < π, x ≠ __, 2 sketch the graph of y = arcsec x and state the range of arcsec x. 161 Chapter 6 Mixed exercise 6 Give any non-exact answers to equations to 1 decimal place. E/P 1 Solve tan x = 2 cot x, in the interval −180° < x < 90°. (4 marks) E/P 2 Given that p = 2 sec θ and q = 4 cos θ, express p in terms of q. (4 marks) E/P 3 Given that p = sin θ and q = 4 cot θ, show that p2q2 = 16(1 − p2). (4 marks) P 4 a Solve, in the interval 0 < θ < 180°, i cosec θ = 2 cot θ ii 2 cot2 θ = 7 cosec θ − 8 i sec(2θ − 15°) = cosec 135° ii sec2 θ + tan θ = 3 b Solve, in the interval 0 < θ < 360°, c Solve, in the interval 0 < x < 2π, __ π 4 i cosec(x + ___) = −√ 2 ii sec2 x = _3 15 E/P P E/P 5 Given that 5 sin x cos y + 4 cos x sin y = 0, and that cot x = 2, find the value of cot y. 6 Prove that: a (tan θ + cot θ)(sin θ + cos θ) ≡ sec θ + cosec θ cosec x b __________ ≡ sec2 x cosec x − sin x c (1 − sin x)(1 + cosec x) ≡ cos x cot x cot x cos x d _________ − _______ ≡ 2 tan x cosec x − 1 1 + sin x 1 1 e __________ + ________ ≡ 2 sec θ tan θ cosec θ − 1 cosec θ + 1 (sec θ − tan θ)(sec θ + tan θ) ≡ cos2 θ f ________ 1 + tan2 θ sin x 1 + cos x 7 a Prove that ________ + ________ ≡ 2 cosec x. 1 + cos x sin x (4 marks) 1 + cos x sin x 4__ b Hence solve, in the interval −2π < x < 2π, _______ + ______ = − ___ 1 + cos x sin x √3 E/P E 1 + cos θ 8 Prove that _______ ≡ (cosec θ + cot θ)2 1 − cos θ (3 marks) __ 3√ 2 b show that cosec A = ____ 4 (3 marks) 10 Given that sec θ = k, |k| > 1, and that θ is obtuse, express in terms of k: a cos θ b tan2 θ (4 marks) (4 marks) π 9 Given that sec A = −3, where __ < A < π, 2 a calculate the exact value of tan A 162 (5 marks) c cot θ d cosec θ Trigonometric functions E π 11 Solve, in the interval 0 < x < 2π, the equation sec(x + __) = 2, giving your answers in terms of π. (5 marks) E/P 12 Find, in terms of π, the value of arcsin(_2) − arcsin(− _2). E/P 13 Solve, in the interval 0 < x < 2π, the equation 1 1 (4 marks) sec2 x giving your answers in terms of π. E/P __ 2√ 3 ___ 3 tan x − 2 = 0, (5 marks) 14 a Factorise sec x cosec x − 2 sec x − cosec x + 2. (2 marks) b Hence solve sec x cosec x − 2 sec x − cosec x + 2 = 0, in the interval 0 < x < 360°. E/P E/P π 15 Given that arctan(x − 2) = − ___ , find the value of x. 3 (4 marks) (3 marks) 16 On the same set of axes sketch the graphs of y = cos x, 0 < x < π, and y = arccos x, −1 < x < 1, showing the coordinates of points at which the curves meet the axes. (4 marks) 17 a Given that sec x + tan x = −3, use the identity 1 + tan2 x ≡ sec2 x to find the value of sec x − tan x. (3 marks) b Deduce the values of: i sec x ii tan x (3 marks) c Hence solve, in the interval −180° < x < 180°, sec x + tan x = −3. (3 marks) E/P 1 18 Given that p = sec θ − tan θ and q = sec θ + tan θ, show that p = __ q (4 marks) E/P 19 a Prove that sec4 θ − tan4 θ ≡ sec2 θ + tan2 θ. (3 marks) b Hence solve, in the interval −180° < θ < 180°, sec4 θ = tan4 θ + 3 tan θ. (4 marks) π _ P 20 a Sketch the graph of y = sin x and shade in the area representing ∫ sin x dx. [illegible]

Remember sin 2θ ≡ 2 sin θ cos θ. Use cos 2A ≡ 2 cos2 A − 1 with A = 2θ. Trigonometry and modelling Example 19 2 Prove the identity tan 2θ ≡ ________ cot θ − tan θ 2 tan θ LHS ≡ tan 2θ ≡ ________ 1 − tan2 θ Divide the numerator and denominator by tan θ. 2 So tan 2θ ≡ ________ 1 ____ − tan θ tan θ Problem-solving Dividing the numerator and denominator by a common term can be helpful when trying to rearrange an expression into a required form. 2 ≡ ________ cot θ − tan θ Example 20 __ π Prove that √ 3 cos 4θ + sin 4θ ≡ 2 cos (4θ − __). 6 π RHS ≡ 2 cos (4θ − __) 6 π π ≡ 2 cos 4θ cos __ + 2 sin 4θ sin __ 6 6 ≡ 2 cos 4θ (1 + 2 sin 4θ (__) 2) 2 __ ≡ √ 3 cos 4θ + sin 4θ ≡ LHS P Sometimes it is easier to begin with the RHS of the identity. __ √3 ___ Exercise Problem-solving Use the addition formulae. π π Write the exact values of cos __ and sin __ 6 6 7F 1 Prove the following identities. cos 2A a ____________ ≡ cos A − sin A cos A + sin A sin B cos B b _____ − _____ ≡ 2 cosec 2A sin (B − A) sin A cos A 1 − cos 2θ c ________ ≡ tan θ sin 2θ sec2 θ ≡ sec 2θ d ________ 1 − tan2 θ e 2(sin3 θ cos θ + cos3 θ sin θ) ≡ sin 2θ sin 3θ cos 3θ f _____ − _____ ≡ 2 cos θ sin θ g cosec θ − 2 cot 2θ cos θ ≡ 2 sin θ sec θ − 1 θ h _______ ≡ tan2 __ 2 sec θ + 1 1 − sin 2x π i tan (__ − x) ≡ __________ 4 cos 2x 187 Chapter 7 P E/P 2 Prove the identities: a sin (A + 60°) + sin (A − 60°) ≡ sin A cos A sin A cos (A + B) b _____ − _____ ≡ ________ sin B cos B sin B cos B sin (x + y) c ________ cos x cos y ≡ tan x + tan y cos (x + y) d ________ + 1 ≡ cot x cot y sin x sin y __ π π e cos (θ + __) + √ 3 sin θ ≡ sin (θ + __) 3 6 cot A cot B − 1 f cot (A + B) ≡ ____________ cot A + cot B g sin2 (45° + θ) + sin2 (45° − θ) ≡ 1 h cos (A + B) cos (A − B) ≡ cos2 A − sin2 B 3 a Show that tan θ + cot θ ≡ 2 cosec 2θ. (3 marks) 4 a Show that sin 3θ ≡ 3 sin θ cos2 θ − sin3 θ. (3 marks) b Hence find the value of tan 75° + cot 75°. E/P (2 marks) b Show that cos 3θ ≡ cos3 θ − 3 sin2 θ cos θ. 3 tan θ − tan3 θ c Hence, or otherwise, show that tan 3θ ≡ ____________ 1 − 3 tan2 θ (3 marks) (4 marks) __ 10√ 2 1 d Given that θ is acute and that cos θ = _3 , show that tan 3θ = _____ 23 (3 marks) 5 a Using cos 2A ≡ 2 cos2 A − 1 ≡ 1 − 2 sin2 A, show that: x 1 + cos x i cos2 __ ≡ ________ 2 2 x 1 − cos x ii sin2 __ ≡ ________ 2 2 b Given that cos θ = 0.6, and that θ is acute, write down the values of: θ θ ii sin __ iii tan __ 2 2 A 1 c Show that cos4 __ ≡ __ (3 + 4 cos A + cos 2A). 2 8 θ i cos __ 2 6 Show that cos4 θ ≡ _8 + _2 cos 2θ + _8 cos 4θ. You must show each stage of your working. (6 marks) E/P 7 Prove that sin2 (x + y) − sin2 (x − y) ≡ sin 2x sin 2y. (5 marks) E/P 8 Prove that cos 2θ − √ 3 sin 2θ ≡ 2 cos (2θ + __). 3 (4 marks) E/P __ π 9 Prove that 4 cos (2θ − __) ≡ 2√ 3 − 4√ 3 sin2 θ + 4 sin θ cos θ. 6 (4 marks) E/P P 3 10 Show that: __ π a cos θ + sin θ ≡ √ 2 sin (θ + __) 4 188 1 π b √ 3 sin 2θ − cos 2θ ≡ 2 sin (2θ − __) 6 __ Trigonometry and modelling Challenge 1 a Show that cos (A + B) − cos (A − B) ≡ −2 sin A sin B. P+Q P−Q b Hence show that cos P − cos Q ≡ −2 sin (______) sin (______). 2 2 c Express 3 sin x sin 7x as the difference of cosines. P+Q P−Q 2 a Prove that sin P + sin Q ≡ 2 sin (______) cos (______). 2 2 __ __ 11π 5π √ 3 + √ 2 b Hence, or otherwise, show that 2 sin ____ cos ___ = _______ 24 24 2 7.7 Modelling with trigonometric functions You can use trigonometric functions to model real-life situations. In trigonometrical modelling questions you will often have to write the model using R sin (x ± α) or R cos (x ± β) to find maximum or minimum values. Example 21 The cabin pressure, P, in pounds per square inch (psi) on an aeroplane at cruising altitude can be modelled by the equation P = 11.5 − 0.5 sin (t − 2), where t is the time in hours since the cruising altitude was first reached, and angles are measured in radians. a State the maximum and the minimum cabin pressure. b Find the time after reaching cruising altitude that the cabin first reaches a maximum pressure. c Calculate the cabin pressure after 5 hours at a cruising altitude. d Find all the times during the first 8 hours of cruising that the cabin pressure would be exactly 11.3 psi. a Maximum pressure = 11.5 − 0.5 × (−1) = 12 psi Minimum pressure = 11.5 − 0.5 × 1 = 11 psi b 11.5 − 0.5 sin (t − 2) = 12 −0.5 sin (t − 2) = 0.5 sin (t − 2) = −1 −1 ≤ sin(t − 2) ≤ 1. Use the maximum and minimum values of the sine function to find the maximum and minimum pressure. Set the model equal to 12, the maximum pressure. π 3π t − 2 = ..., − __, ___, ... 2 2 Remember the model uses radians. t = 0.43 hours = 26 min Multiply 0.43 by 60 to get the time in minutes. c P = 11.5 − 0.5 sin (5 − 2) = 11.5 − 0.070... = 11.43 psi Substitute t = 5. y Online x Explore the solution to this modelling problem graphically using technology. 189 Chapter 7 d 11.5 − 0.5 sin (t − 2) = 11.3 −0.5 sin (t − 2) = −0.2 sin (t − 2) = 0.4 t − 2 = −3.553..., 0.4115..., 2.73..., 6.6947... t = 2.41 hours, 4.73 hours. t = 2 h 25 min, 4 h 44 min Exercise P P Set the model equal to 11.3. Use sin−1(0.4) to find the principal solution, then use the properties of the sine function to find other possible solutions in the range 0 < t < 8. 0 < t < 8 so −2 < t − 2 < 6. There are two solutions in the required range. 7G 1 The height, h, of a buoy on a boating lake can be modelled by h = 0.25 sin (1800t)°, where h is the height in metres above the buoy's resting position and t is the time in minutes. a State the maximum height the buoy reaches above its resting position according to this model. b Calculate the time, to the nearest tenth of a second, at which the buoy is first at a height of 0.1 metres. c Calculate the time interval between successive minimum heights of the buoy. 2 The angle of displacement of a pendulum, θ, at time t seconds after it is released is modelled as θ = 0.03 cos (25t), where all angles are measured in radians. a State the maximum displacement of the pendulum according to this model. b Calculate the angle of displacement of the pendulum after 0.2 seconds. c Find the time taken for the pendulum to return to its starting position. d Find all the times in the first half second of motion that the pendulum has a displacement of 0.01 radians. P 3 The price, P, of stock in pounds during a 9-hour trading window can be modelled by P = 17.4 + 2 sin (0.7t − 3), where t is the time in hours after the stock market opens, and angles are measured in radians. a State the beginning and end price of the stock. b Calculate the maximum price of the stock and the time when it occurs. c A day trader wants to sell the stock when it firsts shows a profit of £0.40 above the day's starting price. At what time should the trader sell the stock? P 4 The temperature of an oven can be modelled by the equation T = 225 − 0.3 sin (2x − 3), where T is the temperate in Celsius and x is the time in minutes after the oven first reaches the desired temperature, and angles are measured in radians. a State the minimum temperature of the oven. b Find the times during the first 10 minutes when the oven is at a minimum temperature. c Calculate the time when the oven first reaches a temperature of 225.2 °C. E/P 5 a Express 0.3 sin θ − 0.4 cos θ in the form R sin (θ − α)°, where R > 0 and 0 < α < 90°. Give the value of α to 2 decimal places. (4 marks) 190 Trigonometry and modelling b i Find the maximum value of 0.3 sin θ − 0.4 cos θ. (2 marks) ii Find the value of θ, for 0 < θ < 180 at which the maximum occurs. (1 mark) Jack models the temperature in his house, T °C, on a particular day by the equation T = 23 + 0.3 sin (18x)° − 0.4 cos (18x)°, x > 0 where x is the number of minutes since the thermostat was adjusted. c Calculate the minimum value of T predicted by this model, and the value of x, to 2 decimal places, when this minimum occurs. (3 marks) d Calculate, to the nearest minute, the times in the first hour when the temperature is predicted, by this model, to be exactly 23 °C. (4 marks) E/P π 6 a Express 65 cos θ − 20 sin θ in the form R cos (θ + α), where R > 0 and 0 < α < __ 2 Give the value of α correct to 4 decimal places. (4 marks) π 7 a Express 200 sin θ − 150 cos θ in the form R sin (θ − α), where R > 0 and 0 < α < __ 2 Give the value of α to 4 decimal places. (4 marks) A city wants to build a large circular wheel as a tourist attraction. The height of a tourist on the circular wheel is modelled by the equation H = 70 − 65 cos 0.2t + 20 sin 0.2t where H is the height of the tourist above the ground in metres, t is the number of minutes after boarding and the angles are given in radians. Find: b the maximum height of the wheel (2 marks) c the time for one complete revolution (2 marks) d the number of minutes the tourist will be over 100 m above the ground in each revolution. (4 marks) E/P The electric field strength, E V/m, in a microwave of width 25 cm can be modelled using the equation 4πx 4πx E = 1700 + 200 sin (____) − 150 cos (____) 25 25 where x is the distance in cm from the left hand edge of the microwave oven. b i Calculate the maximum value of E predicted by this model. ii Find the values of x, for 0 < x , 25, where this maximum occurs. (3 marks) c Food in the microwave will heat best when the electric field strength at the centre of the food is above 1800 V/m. Find the range of possible locations for the centre of the food. (5 marks) Challenge Look at the model for the electric field strength in a microwave oven given in question 7 above. For food of the same type and mass, the energy transferred by the oven is proportional to the square of the electric field strength. Given that a square of chocolate placed at a point of maximum field strength takes 20 seconds to melt, a estimate the range of locations within the oven that an identical square of chocolate will take longer than 30 seconds to melt. b State two limitations of the model. 191 Chapter 7 Mixed exercise P 7 1 a Without using a calculator, find the value of: 1__ 1__ cos 15° − ___ sin 15° i sin 40° cos 10° − cos 40° sin 10° ii ___ √2 √2 1 − tan 15° iii _________ 1 + tan 15° __ P P √5 + 1 1__ where x is acute and that cos (x − y) = sin y, show that tan y = _____ 2 Given that sin x = ___ 2 √5 3 The lines l1 and l2 , with equations y = 2x and 3y = x − 1 respectively, are drawn on the same set of axes. Given that the scales are the same on both axes and that the angles l1 and l2 make with the positive x-axis are A and B respectively, a write down the value of tan A and the value of tan B; b without using your calculator, work out the acute angle between l1 and l2. P P P 4 In △ABC, AB = 5 cm and__AC = 4 cm, ∠ABC = (θ − 30°) and ∠ACB = (θ + 30°). Using the sine rule, show that tan θ = 3√ 3 . __ 5 The first three terms of an arithmetic series are √ 3 cos θ, sin (θ − 30°) and sin θ, where θ is acute. Find the value of θ. 6 Two of the angles, A and B, in △ABC are such that tan A = _4 , tan B = 12 3 a Find the exact value of: i sin (A + B) 5 ii tan 2B. b By writing C as 180° − (A + B), show that cos C = − __ 65 P 33 3 2___ and cos y = ____ 7 The angles x and y are acute angles such that sin x = ___ √5 √ 10 _3 a Show that cos 2x = − 5 b Find the value of cos 2y. c Show without using your calculator, that: π i tan (x + y) = 7 ii x − y = __ 4 P 8 Given that sin x cos y = _2 and cos x sin y = _3 , 1 1 a show that sin (x + y) = 5 sin (x − y). Given also that tan y = k, express in terms of k: b tan x c tan 2x E/P __ 1__ 9 a Given that √ 3 sin 2θ + 2 sin2 θ = 1, show that tan 2θ = ___ √3 __ b Hence solve, for 0 < θ < π, the equation √ 3 sin 2θ + 2 sin2 θ = 1. 192 (2 marks) (4 marks) Trigonometry and modelling E/P E/P 10 a Show that cos 2θ = 5 sin θ may be written in the form a sin2 θ + b sin θ + c = 0, where a, b and c are constants to be found. (3 marks) b Hence solve, for −π < θ < π, the equation cos 2θ = 5 sin θ. (4 marks) 1 __ 11 a Given that 2 sin x = cos (x − 60)°, show that tan x = _____ 4 − √3 (4 marks) b Hence solve, for 0 < x < 360°, 2 sin x = cos (x − 60°), giving your answers to 1 decimal place. E/P 12 a Given that 4 sin (x + 70°) = cos (x + 20°), show that tan x = − _5 tan 70°. 3 b Hence solve, for 0 < x < 180°, 4 sin (x + 70°) = cos (x + 20°), giving your answers to 1 decimal place. P (2 marks) (4 marks) (3 marks) 13 a Given that α is acute and tan α = _4 , prove that 3 3 sin (θ + α) + 4 cos (θ + α) ≡ 5 cos θ b Given that sin x = 0.6 and cos x = −0.8, evaluate cos (x + 270°) and cos (x + 540°). E/P 14 a Prove, by counter-example, that the statement sec (A + B) ≡ sec A + sec B, for all A and B is false. nπ b Prove that tan θ + cot θ ≡ 2 cosec 2θ, θ ± ___ , n ∈ . 2 P E/P (2 marks) (4 marks) 2 tan θ with an appropriate value of θ, 15 Using tan 2θ ≡ ________ 1 − tan2 θ __ π a show that tan ___ = √ 2 − 1. 8 3π b Use the result in a to find the exact value of tan ___ 8 __ 16 a Express sin x − √ 3 cos x in the form R sin (x − α), with R > 0 and 0 < α < 90°. __ (4 marks) b Hence sketch the graph of y = sin x − √ 3 cos x, for −360° < x < 360°, giving the coordinates of all points of intersection with the axes. (4 marks) E/P π 17 Given that 7 cos 2θ + 24 sin 2θ ≡ R cos (2θ − α), where R > 0 and 0 < α < __, find: 2 a the value of R and the value of α, to 2 decimal places b the maximum value of 14 cos2 θ + 48 sin θ cos θ. (4 marks) (1 mark) c Solve the equation 7 cos 2θ + 24 sin 2θ = 12.5, for 0 < θ < 180°, giving your answers to 1 decimal place. (5 marks) 193 Chapter 7 E/P π 18 a Express 1.5 sin 2x + 2 cos 2x in the form R sin (2x + α), where R > 0 and 0 < α < __ , 2 giving your values of R and α to 3 decimal places where appropriate. (4 marks) b Express 3 sin x cos x + 4 cos2 x in the form a sin 2x + b cos 2x + c, where a, b and c are constants to be found. (3 marks) c Hence, using your answer to part a, deduce the maximum value of 3 sin x cos x + 4 cos2 x. E/P E/P (1 mark) ___ θ 19 a Given that sin2 __ = 2 sin θ, show that √ 17 sin (θ + α) = 1 and state the value of α. 2 θ b Hence, or otherwise, solve sin2 __ = 2 sin θ for 0 < θ < 360°. 2 (4 marks) 21 Using known trigonometric identities, prove the following: π π a sec θ cosec θ ≡ 2 cosec 2θ b tan (__ + x) − tan (__ − x) ≡ 2 tan 2x 4 4 c sin (x + y) sin (x − y) ≡ cos2 y − cos2 x E/P (4 marks) 20 a Given that 2 cos θ = 1 + 3 sin θ, show that R cos (θ + α) = 1, where R and α are constants to be found. (2 marks) b Hence, or otherwise, solve 2 cos θ = 1 + 3 sin θ for 0 < θ < 360°. P (3 marks) d 1 + 2 cos 2θ + cos 4θ ≡ 4 cos2 θ cos 2θ 1 − cos 2x 22 a Use the double-angle formulae to prove that ________ ≡ tan2 x. 1 + cos 2x (4 marks) 1 − cos 2x b Hence find, for −π < x < π, all the solutions of ________ = 3, leaving your answers 1 + cos 2x in terms of π. (2 marks) E/P 23 a Prove that cos4 2θ − sin4 2θ ≡ cos 4θ. (4 marks) _1 b Hence find, for 0 < θ < 180°, all the solutions of cos4 2θ − sin4 2θ = 2 E/P (2 marks) 1 − cos 2θ 24 a Prove that ________ ≡ tan θ. sin 2θ (4 marks) b Verify that θ = 180° is a solution of the equation sin 2θ = 2 − 2 cos 2θ. (1 mark) c Using the result in part a, or otherwise, find the two other solutions, 0 < θ < 360°, of the equation sin 2θ = 2 − 2 cos 2θ. (3 marks) E/P __ 25 The curve on an oscilloscope screen satisfies the equation y = 2 cos x − √ 5 sin x. a Express the equation of the curve in the form y = R cos (x + α), where R and π α are constants and R . 0 and 0 < α , __ 2 b Find the values of x, 0 < x , 2π, for which y = −1. 194 (4 marks) (3 marks) Trigonometry and modelling E/P 26 a Express 1.4 sin θ − 5.6 cos θ in the form R sin (θ − α), where R and α are constants, R > 0 and 0 < α < 90°. Round R and α to 3 decimal places. (4 marks) b Hence find the maximum value of 1.4 sin θ − 5.6 cos θ and the smallest positive value of θ for which this maximum occurs. (3 marks) The length of daylight, d(t) at a location in northern Scotland can be modelled using the equation 360t ° 360t ° d(t) = 12 + 5.6 cos (____) + 1.4 sin (____) 365 365 where t is the numbers of days into the year. c Calculate the minimum number of daylight hours in northern Scotland as given by this model. (2 marks) d Find the value of t when this minimum number of daylight hours occurs. E/P 27 a Express 12 sin x + 5 cos x in the form R sin (x + α), where R and α are constants, R > 0 and 0 < α < 90°. Round α to 1 decimal place. A runner's speed, v in m/s, in an endurance race can be modelled by the equation 50 v (x) = ________________________ , 0 < x < 300 2x ° 2x ° 12 sin (___) + 5 cos (___) 5 5 where x is the time in minutes since the beginning of the race. b Find the minimum value of v. c Find the time into the race when this speed occurs. (1 mark) (4 marks) (2 marks) (1 mark) Challenge 1 Prove the identities: cos 2θ + cos 4θ a ____________ ≡ −cot θ sin 2θ − sin 4θ b cos x + 2 cos 3x + cos 5x ≡ 4 cos2 x cos 3x 2 The points A, B and C lie on a circle with centre O and radius 1. AC is a diameter of the circle and point D lies on OC such that ∠ODB = 90°. B A θ 1 O D Hint Find expressions for ∠BOD and AB, then consider the lengths OD and DB. C Use this construction to prove: a sin 2θ ≡ 2 sin θ cos θ b cos 2θ ≡ 2 cos2 θ − 1 195 Chapter 7 Summary of key points 1 The addition (or compound-angle) formulae are: • sin (A + B) ≡ sin A cos B + cos A sin B sin (A − B) ≡ sin A cos B − cos A sin B tan A + tan B • tan (A + B) ≡ ____________ 1 − tan A tan B tan A − tan B tan (A − B) ≡ ____________ 1 + tan A tan B • cos (A + B) ≡ cos A cos B − sin A sin B cos (A − B) ≡ cos A cos B + sin A sin B 2 The double-angle formulae are: • sin 2A ≡ 2 sin A cos A • cos 2A ≡ cos2 A − sin2 A ≡ 2 cos2 A − 1 ≡ 1 − 2 sin2 A 2 tan A • tan 2A ≡ _________ 1 − tan2 A 3 You can write any expression of the form a sin θ + b cos θ as either: • R sin (θ ± a), with R > 0 and 0 < α < 90° or • R cos (θ ± b), with R > 0 and 0 < α < 90° _____ where R cos α = a, R sin α = b and R = √ a2 + b2 . 196 Parametric equations Objectives After completing this chapter you should be able to: 8 ● Convert parametric equations into Cartesian form by substitution → pages 198–202 ● Convert parametric equations into Cartesian form using → pages 202–206 trigonometric identities ● Understand and use parametric equations of curves and sketch → pages 206–208 parametric curves ● Solve coordinate geometry problems involving parametric → pages 209–213 equations ● Use parametric equations in modelling in a variety of contexts → pages 213–220 Prior knowledge check 1 Rearrange to make t the subject: c y = 2 − 4 ln t a x = 4t − kt b y = 3t2 d x = 1 + 2e−3t ← GCSE Mathematics; Year 1, Chapter 14 2 3 Parametric equations can be used to describe the path of a ski jumper from the point of leaving the ski ramp to the point of landing. → Exercise 8E, Q8 Write in terms of powers of cos x: b sin 2x a 4 + 3 sin2 x c cot x d 2 cos x + cos 2x ← Section 7.2 State the ranges of the following functions. a y = ln (x + 1), x > 0 b y = 2 sin x, 0 < x < π 1 d y = ______ , x > −2 c y = x2 + 4x − 2, −4 < x < 1 2x + 5 ← Year 1, Chapter 6 4 A circle has centre (0, 4) and radius 5. Find the coordinates of the points of intersection of the circle and the line with equation ← Year 1, Chapter 6 2y – x – 10 = 0. 197 Chapter 8 8.1 Parametric equations You can write the x- and y-coordinates of each point on a curve as functions of a third variable. This variable is called a parameter and is often represented by the letter t. ■ A curve can be defined using parametric equations x = p(t) and y = q(t). Each value of the parameter, t, defines a point on the curve with coordinates (p(t), q(t)). y 6 2 x = t + 1 , y = 2t, t > 0 t 5 4 3 2 1 –1 O –1 1 2 3 4 5 6 x Watch out The value of the parameter is generally not equal to either the x- or the y-coordinate, and more than one point on a curve can have the same x-coordinate. ■ You can convert between parametric equations and Cartesian equations by using substitution to eliminate the parameter. These are the parametric equations of the curve. The domain of the parameter tells you the values of t you would need to substitute to find the coordinates of the points on the curve. 22 + 1 When t = 2, x = _____ = 2.5 and y = 2 × 2 = 4. 2 This corresponds to the point (2.5, 4). 0.52 + 1 When t = 0.5, x = _______ = 2.5 and y = 2 × 0.5 = 1. 0.5 This corresponds to the point (2.5, 1). Notation A Cartesian equation in two dimensions involves the variables x and y only. You can use the domain and range of the parametric functions to find the domain and range of the resulting Cartesian function. ■ For parametric equations x = p(t) and y = q(t) with Cartesian equation y = f(x): the domain of f(x) is the range of p(t) ● the range of f(x) is the range of q(t) ● Example 1 A curve has parametric equations x = 2t, y = t2, −3 < t < 3 a Find a Cartesian equation of the curve in the form y = f(x). b State the domain and range of f(x). c Sketch the curve within the given domain for t. x a x = 2t so t = __ 2 y = t2 (1) (2) Substitute (1) into (2): x 2 x2 y = (__) = ___ 2 4 198 A Cartesian equation only involves the variables x and y, so you need to eliminate t. Rearrange one equation into the form t =... then substitute into the other equation. This is a quadratic curve. Parametric equations b x = 2t, −3 , t , 3 So the domain of f(x) is −6 , x , 6. y = t2, −3 , t , 3 So the range of f(x) is 0 < y , 9. c y x = 2t y = t2 9 The domain of f is the range of the parametric function for x. The range of x = 2t over the domain −3 < t < 3 is −6 < x < 6. ← Section 2.1 The range of f is the range of the parametric function for y. Choose your inequalities carefully. y = t2 can equal 0 in the interval −3 < t < 3, so use k where k is a constant to be found. x = ln (t + 3), b Write down the range of f(x). a x = ln (t + 3) ex = t + 3 So ex − 3 = t Substitute t = ex − 3 into 1 1 y = _____ t + 5 ex − 3 + 5 1 = _____ ex + 2 When t = −2: x = ln (t + 3) = ln 1 = 0 As t increases ln (t + 3) increases, so the range of the parametric function for x is x > 0. The Cartesian equation is 1 y = ______ ,x>0 ex + 2 1 1 b When t = −2: y = _____ = __ t+5 3 As t increases y decreases, so the range 1 of the parametric function for y is y < __ 3 1 The range of f(x) is y < __ 3 y Online x Sketch this parametric curve using technology. ex is the inverse function of ln x. Rearrange the equation for x into the form t = ... then substitute into the equation for y. To find the domain for f(x), consider the range of values x can take for values of t > −2. You need to consider what value x takes when t = −2 and what happens when t increases. The range of f is the range of values y can take within the given range of the parameter. You could also find the range of f(x) by considering the domain of f(x). f(0) = _13 and f(x) decreases as x increases, so the range of f(x) is y < _13 ← Section 2.1 199 Chapter 8 Exercise 8A 1 Find a Cartesian equation for each of these parametric equations, giving your answer in the form y = f(x). In each case find the domain and range of f(x). a x = t − 2, y = t2 + 1, 1 c x = __t , y = 3 − t, d x = 2t + 1, 1 f x = _____ , t+1 t∈ℝ If the domain of t is given as t ≠ 0, this implies that t can take any value in other than 0. 1 e x = _____ , t−2 t>0 1 y = _____ , t−2 y = t2 − 1, Notation t≠0 1 y = __t , b x = 5 − t, −4 < t < 4 y = t2, t>2 t>2 2 For each of these parametric curves: i find a Cartesian equation for the curve in the form y = f(x) giving the domain on which the curve is defined ii find the range of f(x). 1 b x = ln (t + 3), y = _____ , t > −2 a x = 2 ln (5 − t), y = t2 − 5, t < 4 t+5 c x = et, P y = e3t, t∈ℝ _3 A curve C is defined by the parametric equations x = √ t , y = t(9 − t), 0 < t < 5. a Find a Cartesian equation of the curve in the form y = f(x), and determine the domain and range of f(x). b Sketch C showing clearly the Problem-solving coordinates of any turning y = t(9 − t) is a quadratic with a negative t2 term and roots at points, endpoints and intersections t = 0 and t = 9. It will take its maximum value when t = 4.5. with the coordinate axes. 4 For each of the following parametric curves: i find a Cartesian equation for the curve in the form y = f(x) ii find the domain and range of f(x) iii sketch the curve within the given domain of t. a x = 2t2 − 3, c x = t + 1, y = 9 − t2, 1 y = _____ , t−1 e x = ln (4 − t), 200 t ∈ ℝ, y = t − 2, b x = 3t − 1, y = (t − 1)(t + 2), −4 < t < 4 t>0 t0 Parametric equations P 5 The curves C1 and C2 are defined by the following parametric equations. t 1 C2: x = ______ , y = ______ C1: x = 1 + 2t, y = 2 + 3t 2 < t < 5 2t − 3 2t − 3 a Show that both curves are segments of the same straight line. Notation Straight lines and line segments can be referred to as 'curves' in coordinate geometry. b Find the length of each line segment. E/P 2 k where k is a constant to be found. b Write down the range of f(x). E/P 8 A diagram shows a curve C with parametric equations _ y = t3 − 2t, x = 3√ t , 0 0. There is only one point of intersection so choose any of these values of t. You can use one of the 2π 8π others to check: 8 cos (20 × ___) = 8 cos (___) = −4 15 3 Parametric equations c The figure skater crosses the y-axis when x = 0, 0 = 8 cos t 0 = cos t π 3π 5π 7π So ___ , ___ , ___ , ___ , ... 2 2 2 2 Substitute these t-values into y. π t = __ : 2 π π π 1 y = 12 sin (__ × __ − __) = 12 sin (− __) 2 2 3 12 = −3.11 (2 d.p.) 3π t = ___ : 2 5π 1 3π π y = 12 sin (__ × ___ − __) = 12 sin (___) 2 2 12 3 = 11.59 (2 d.p.) 5π t = ___ : 2 11π 1 5π π y = 12 sin (__ × ___ − __) = 12 sin(___) 2 2 12 3 = 3.11 (2 d.p.) 7π t = ___ : 2 17π 1 7π π y = 12 sin (__ × ___ − __) = 12 sin(___) 2 2 12 3 = 11.59 (2 d.p.) So the skater crosses the y-axis at (0, −3.11), (0, 11.59), (0, 3.11), (0, −11.59). 2π d The period of x = 8 cos 20t is ___ , 20 so the skater returns to his x-position 4π 2π after ___ min, ___ min, ... 20 20 2π π The period of y = 12 sin (10t − __) is ___, 3 10 so the skater returns to his y-position 4π 2π after ___ min, ___ min, ... 10 10 So the skater first completes a full figure-of-eight motion after 2π ___ mins = 0.628 ... mins or 38 seconds 10 (2 s.f.). Find solutions to 8 cos t = 0 in the domain t > 0. There are 4 points of intersection so consider the first 4 solutions, and check that these give different values of y. Use your calculator to find the corresponding values of y. You can give your answers as decimals __ __ π or as exact values: 12 sin (− ___) = −3√ 6 + 3√ 6 12 y Online x Find points of intersection of this curve with the coordinate axes using technology. Check that these look sensible from the graph. The motion of the skater appears to be symmetrical about the x-axis so these look right. 2π The period of a cos (bx + c) is ___ and the period b 2π of a sin (bx + c) is ___ b Problem-solving In order for the figure skater to return to his starting position, both parametric equations must complete full periods. This occurs at the least common multiple of the two periods. 217 Chapter 8 Exercise P 8E 1 A river flows from north to south. The position at time t seconds of a rowing boat crossing the river from west to east is modelled by the parametric equations x = 0.9 t m, y = −3.2 t m where x is the distance travelled east and y is the distance travelled north. Given that the river is 75 m wide, a find the time taken to get to the other side b find the distance the boat has been moved off-course due to the current c show that the motion of the boat is a straight line d determine the speed of the boat. P 2 The position of a small plane coming into land at time t minutes after it has started its descent is modelled by the parametric equations x = 80t, y = −9.1t + 3000, 0 < t < 329 where x is the horizontal distance travelled (in metres) and y is the vertical distance travelled (in metres) from the point of starting its descent. a Find the initial height of the plane. b Justify the choice of domain, 0 < t < 329, for this model. c Find the horizontal distance the plane travels between beginning its descent and landing. P 3 A ball is kicked from the ground with an initial speed of 20 m s−1 at an angle of 30°. Its position after t seconds can be described using the following parametric equations __ x = 10√ 3 t m, y = (−4.9t2 + 10t) m, 0 < t < k a Find the horizontal distance travelled by the ball when it hits the ground. A player wants to head the ball when it is descending between 1.5 m and 2.5 m off the ground. b Find the range of time after the ball has been kicked at which the player can head the ball. c Find the closest distance from where the ball has been kicked at which the player can head the ball. P 4 The path of a dolphin leaping out of the water can be modelled with the following parametric equations x = 2t m, y = −4.9t2 + 10t m where x is the horizontal distance from the point the dolphin jumps out of the water, y is the height above sea level of the dolphin and t is the time in seconds after the dolphin has started its jump. a Find the time the dolphin takes to complete a single jump. b Find the horizontal distance the dolphin travels during a single jump. c Show that the dolphin's path is modelled by a quadratic curve. d Find the maximum height of the dolphin. 218 Parametric equations P 5 The path of a car on a Ferris wheel at time t minutes is modelled using the parametric equations x = 12 sin t, y = 12 − 12 cos t where x is the horizontal distance in metres of the car from the start of the ride and y is the height in metres above ground level of the car. a Show that the motion of the car is a circle with radius 12 m. b Hence, find the maximum height of the car during the journey. c Find the time taken to complete one revolution of the Ferris wheel and hence calculate the average speed of the car. E/P 6 The cross-section of a bowl design is given by the following parametric equations y π π x = t − 4 sin t, y = 1 − 2 cos t, − __ < t < __ 2 2 P a Find the length of the opening of the bowl. (3 marks) b Given that the cross-section of the bowl crosses the y-axis at its deepest point, find the depth of the bowl. (4 marks) 7 A particle is moving in the xy-plane such that its position after time t seconds relative to a fixed origin O is given by the parametric equations t2 − 3t + 2 , x = ________ t y = 2t, x O y t>0 The diagram shows the path of the particle. a Find the distance from the origin to the particle at time t = 0.5. b Find the coordinates of the points where the particle crosses the y-axis. x O Another particle travels in the same plane with its path given by the equation y = 2x + 10. c Show that the paths of these two particles never intersect. E/P 8 The path of a ski jumper from the point of leaving the ramp to the point of landing is modelled using the parametric equations x = 18t, y = −4.9t2 + 4t + 10, 0 k where k is a constant to be found in exact form. b Write down the range of f(x), leaving your answer in exact form. 4 A curve has parametric equations 1 x = ______ , 2t + 1 1 y = 2 ln (t + __), 2 1 t > __ 2 Find a Cartesian equation of the curve in the form y = f(x), and state the domain and range of f(x). P 5 A curve has parametric equations x = sin t, y = cos 2t, 0 < t , 2π a Find a Cartesian equation of the curve. The curve cuts the x-axis at (a, 0) and (b, 0). b Find the values of a and b. P 1 6 A curve has parametric equations x = _____ , 1+t 1 y = __________ , (1 + t)(1 − t) t>1 x2 Express t in terms of x, and hence show that a Cartesian equation of the curve is y = ______ 2x − 1 7 A circle has parametric equations x = 4 sin t − 3, y = 4 cos t + 5, 0 < t < 2π a Find a Cartesian equation of the circle. b Draw a sketch of the circle. c Find the exact coordinates of the points of intersection of the circle with the y-axis. E/P 8 The curve C has parametric equations 2 − 3t x = ______ , 1+t E 3 + 2t y = ______ , 1+t 0 0 for every value of x in that interval. Links d2y To find the second derivative, f0(x) or ___2 , you differentiate twice with respect to x. dx ← Year 1, Chapter 12 y y O y y = ex y = 5x − x2 y = x3 − 6x2 − 9x x O O dy ____ = −2 so the curve is 2 2 dx concave for all x ∈ . 2 x x dy ____ = ex which is always 2 dy ____ = 6x − 12 so the curve is 2 2 2 dx positive, so the curve is convex for all x ∈ . dx concave for all x < 2 and convex for x > 2. Example 26 Find the interval on which the function f(x) = x3 + 4x + 3 is concave. f(x) = x3 + 4x + 3 f9(x) = 3x2 + 4 f0(x) = 6x For f(x) to be concave, f0(x) < 0 6x < 0 x < 0 for all x ∈ , so + 2 > 2 for all x∈ Hence f0(x) > 0, so f is convex for all x ∈ . e2x 4e2x Problem-solving Write down the condition for a convex function and a conclusion. The point at which a curve changes from being concave to convex (or vice versa) is called a point of inflection. The diagram shows the curve with equation y = x3 − 2x2 − 4x + 5. y In the interval [−2, 0] the curve is concave. 5 –2 –1 O 1 2 3 x In the interval [1, 3], the curve is convex. –5 At some point between 0 and 1 the curve changes from being concave to being convex. This is the point of inflection. ■ A point of inflection is a point at which f 0(x) changes sign. To find a point of inflection you need to show that f 0(x) = 0 at that point, and that it has different signs on either side of that point. Watch out A point of inflection does not have to be a stationary point. Example 28 The curve C has equation y = x3 − 2x2 − 4x + 5. a Show that C is concave on the interval [−2, 0] and convex on the interval [1, 3]. b Find the coordinates of the point of inflection. a dy ___ dx = d y ___ 3x 2 − 4x − 4 Differentiate y = x3 − 2x2 − 4x + 5 with respect to x twice. 2 = 6x − 4 dx 2 d 2y ____ = 6x − 4 < 0 for all −2 < x < 0. dx 2 258 Consider the value of 6x − 4 on the interval [−2, 0]. 6x − 4 is a linear function. When x = −2, d 2y d 2y d 2y ____ ____ ____ = −16 and when x = 0, = −4, so 0 for all 1 < x < 3. Therefore, y = x3 − 2x2 − 4x + 5 is convex on the interval [1, 3]. d 2y b ____2 = 6x − 4 = 0 dx Find the point where f0(x) = 0. You have already determined that f0(x) changes sign on either side of this point. 6x = 4 4 __ 2 x = __ 6 = 3 Substitute x into y gives y y = (3) − 2 (3) − 4(3) + 5 = __ 2 3 __ 2 2 __ 2 47 ___ Online x Explore the solution to this example graphically using technology. 27 So the point of inflection of the curve C 47 2 ___ is (__ 3 , 27). Exercise P 9I 1 For each of the following functions, find the interval on which the function is: i convex P ii concave a f(x) = x3 − 3x2 + x − 2 b f(x) = x4 − 3x3 + 2x − 1 c f(x) = sin x, 0 < x < 2π d f(x) = e f(x) = f f(x) = ln x, x > 0 −x2 + 3x − 7 ex − x2 f(x) = arcsin x, −1 < x < 1 a Show that f(x) is concave on the interval (−1, 0). b Show that f(x) is convex on the interval (0, 1). c Hence deduce the point of inflection of f. P 3 Find any point(s) of inflection of the following functions. x 3 − 2x 2 + x − 1 a f(x) = cos2 x − 2 sin x, 0 < x < 2π b f(x) = − _____________ , x ≠ 2 x−2 3 x , x ≠ 2 d f(x) = arctan x c f(x) = − _____ x2 − 4 P 4 f(x) = 2x2 ln x, x > 0 Show that f has exactly one point of inflection and determine the value of x at this point. P 5 The curve C has equation y = ex(x2 − 2x + 2). a Find the exact coordinates of the stationary point on C and determine its nature. b Find the coordinates of any non-stationary points of inflection on C. 259 Chapter 9 P 6 The curve C has equation y = xex. a Find the exact coordinates of the stationary point on C and determine its nature. Problem-solving b Find the coordinates of any non-stationary points of inflection on C. Consider how C behaves for very large positive and negative values of x. c Hence sketch the graph of y = xex. P 7 For each point on the graph, state whether: i f9(x) is positive, negative or zero ii f0(x) is positive, negative or zero y y = f(x) A C D O x B P E E/P π π 8 f(x) = tan x, − __ < x < __ 2 2 Prove that f(x) has exactly one point of inflection, at the origin. 9 Given that y = x(3x − 1)5, dy d 2y a find ___ and ___2 dx dx (4 marks) b find the points of inflection of y. (4 marks) 10 A student is attempting to find the points of inflection on the curve C with equation y = (x − 5)4 The attempt is shown below: dy ___ = 4(x − 5) 3 dx d2y ____ = 12 (x − 5) 2 When dx2 2y d ____ dx2 12(x − 5)2 (x − 5)2 x−5 x = 0, = = = = 0 0 0 5 Therefore, the curve C has a point of inflection at x = 5. 260 a Identify the mistake made by the student. (2 marks) b Write down the coordinates of the stationary point on C and determine its nature. (2 marks) Differentiation E/P 11 A curve C has equation y = _3 x 2 ln x − 2x + 5, x > 0 1 _3 (5 marks) Show that the curve C is convex for all x > e − 2. Challenge 1 Prove that every cubic curve has exactly one point of inflection. 2 The curve C has equation y = ax4 + bx 3 + cx2 + dx + e, a ≠ 0 a Show that C has at most two points of inflection. b Prove that if 3b2 < 8ac, then C has no points of inflection. 9.10 Rates of change ■ You can use the chain rule to connect rates of change in situations involving more than two variables. Example 29 Given that the area of a circle A cm2 is related to its radius r cm by the formula A = πr2, and that dr dA the rate of change of its radius in cm s−1 is given by ___ = 5, find ___ when r = 3. dt dt A = πr2 ∴ dA ___ = dr dA Using ___ = dt dA ___ = dt Problem-solving 2πr dA ___ dr dr × ___ dt 2πr × 5 = 30π, when r = 3. In order to be able to apply the chain rule to dA dA find ___ you need to know ___. You can find it by dt dr differentiating A = πr2 with respect to r. You should use the chain rule, giving the derivative which you need to find in terms of known derivatives. Example 30 The volume of a hemisphere V cm3 is related to its radius r cm by the formula V = _3 πr3 and the total surface area S cm2 is given by the formula S = πr2 + 2πr2 = 3πr2. Given that the rate of dV dS increase of volume, in cm3 s−1, ___ = 6, find the rate of increase of surface area ___ dt dt 2 2 3 2 V = __ 3 πr and S = 3πr dV ___ dr dS = 2πr2 and ___ = 6πr dr This is area of circular base plus area of curved surface. dV dS As V and S are functions of r, find ___ and ___ dr dr 261 Chapter 9 dS dS dr dV Now ___ = ___ × ___ × ___ dt dr dV dt 1 = 6πr × _____2 × 6 2πr Use the chain rule together with the property dV dr that ___ = 1 ÷ ___ dV dr 18 = ___ r An equation which involves a rate of change is called a differential equation. You can formulate differential equations from information given in a question. Links You can use integration to solve differential equations. → Section 11.10 Example 31 In the decay of radioactive particles, the rate at which particles decay is proportional to the number of particles remaining. Write down a differential equation for the rate of change of the number of particles. Let N be the number of particles and let t be time. The rate of change of the number of dN particles ___ is proportional to N. dt dN i.e. ___ = −kN, where k is a positive constant. dt The minus sign arises because the number of particles is decreasing. dN dN ___ ∝ N so you can write ___ = kN dt dt where k is the constant of proportion. Example 32 Newton's law of cooling states that the rate of loss of temperature of a body is proportional to the excess temperature of the body over its surroundings. Write an equation that expresses this law. Let the temperature of the body be θ degrees and the time be t seconds. [illegible]

equation x2 + ex − 4 = 0. (1 mark) _1 c Show that the equation x2 + ex − 4 = 0 can be written in the form x = ±(4 − ex) 2 (2 marks) _1 2 The iterative formula xn + 1 = −(4 − exn) , x0 = −2, is used to find an approximate value for the negative root. d Calculate the values of x1, x2, x3 and x4 to 4 decimal places. (3 marks) e Explain why the starting value x0 = 1.4 will not produce a valid result with this formula. (2 marks) 289 Chapter 10 E/P 4 g(x) = x5 − 5x − 6 a Show that g(x) = 0 has a root, α, between x = 1 and x = 2. (2 marks) _1 b Show that the equation g(x) = 0 can be written as x = (px + q) r , where p, q and r are integers to be found. (2 marks) r The iterative formula xn + 1 = (pxn + q) r , x0 = 1 is used to find an approximate value for α. _1 c Calculate the values of x1, x2 and x3 to 4 decimal places. (3 marks) d By choosing a suitable interval, show that α = 1.708 is correct to 3 decimal places. (3 marks) E/P 5 g(x) = x2 − 3x − 5 _____ a Show that the equation g(x) = 0 can be written as x = √ 3x + 5 . (1 mark) _____ b Sketch on the same axes the graphs of y = x and y = √ 3x + 5 . _____ (2 marks) c Use your diagram to explain why the iterative formula xn + 1 = √ 3xn + 5 converges (1 mark) to a root of g(x) when x0 = 1. 2−5 x n g(x) = 0 can also be rearranged to form the iterative formula xn + 1 = _____ 3 d With reference to a diagram, explain why this iterative formula diverges when x0 = 7. (3 marks) E/P 6 f(x) = 5x − 4 sin x − 2, where x is in radians. a Show that f(x) = 0 has a root, α, between x = 1.1 and x = 1.15. b Show that f(x) = 0 can be written as x = p sin x + q , where p and q are rational numbers to be found. (2 marks) (2 marks) c Starting with x0 = 1.1, use the iterative formula xn + 1 = p sin xn + q with your values (3 marks) of p and q to calculate the values of x1, x2, x3 and x4 to 3 decimal places. E/P 1 7 a On the same axes, sketch the graphs of y = __ x and y = x + 3. (2 marks) 1 b Write down the number of roots of the equation __ x = x + 3. (1 mark) 1 c Show that the positive root of the equation __ x = x + 3 lies in the interval (0.30, 0.31). 1 2 d Show that the equation __ x = x + 3 may be written in the form x + 3x − 1 = 0. (2 marks) (2 marks) e Use the quadratic formula to find the positive root of the equation x2 + 3x − 1 = 0 to 3 decimal places. (2 marks) E/P 8 g(x) = x3 − 7x2 + 2x + 4 a Find g9(x). A root α of the equation g(x) = 0 lies in the interval [6.5, 6.7]. (2 marks) b Taking 6.6 as a first approximation to α, apply the Newton–Raphson process once to g(x) to obtain a second approximation to α. Give your answer to 3 decimal places. (4 marks) 290 Numerical methods E/P c Given that g(1) = 0, find the exact value of the other two roots of g(x). (3 marks) d Calculate the percentage error of your answer in part b. (2 marks) π n 9 f(x) = 2 sec x + 2x − 3, − _ < x < _ where x is in radians. 2 a Show that f(x) = 0 has a solution, α, in the interval 0.4 < x < 0.5. (2 marks) b Taking 0.4 as a first approximation to α, apply the Newton–Raphson process once to f(x) to obtain a second approximation to α. Give your answer to 3 decimal places. (4 marks) c Show that x = −1.190 is a different solution, β, of f(x) = 0 correct to 3 decimal places. (2 marks) E/P 3 1 10 f(x) = e0.8x − _____ , x ≠ __ 3 − 2x 2 (3 marks) a Show that the equation f(x) = 0 can be written as x = 1.5 − 0.5e−0.8x. b Use the iterative formula xn + 1 = 1.5 − with x0 = 1.3 to obtain x1, x2 and x3. Hence write down one root of f(x) = 0 correct to 3 decimal places. (2 marks) 0.5e−0.8x c Show that the equation f(x) = 0 can be written in the form x = p ln (3 − 2x), stating the value of p. (3 marks) d Use the iterative formula xn + 1 = p ln (3 − 2xn) with x0 = −2.6 and the value of p found in part c to obtain x1, x2 and x3. Hence write down a second root of f(x) = 0 correct to 2 decimal places. (2 marks) E/P dy 11 a By writing y = xx in the form ln y = x ln x, show that ___ = xx (ln x + 1). dx x b Show that the function f(x) = x − 2 has a root, α, in the interval [1.4, 1.6]. (4 marks) (2 marks) c Taking x0 = 1.5 as a first approximation to α, apply the Newton–Raphson procedure once to obtain a second approximation to α, giving your answer to 4 decimal places. (4 marks) d By considering a change of sign of f(x) over a suitable interval, show that α = 1.5596, correct to 4 decimal places. (3 marks) E/P 12 The diagram shows part of the curve with equation 1 y = f(x), where f(x) = cos (4x) − _2 x. y 2 a Show that the curve has a root in the interval [1.3, 1.4]. (2 marks) 1 b Use differentiation to find the coordinates of point B. Write each coordinate correct to 3 decimal places. (3 marks) O 1 1 c Using the iterative formula xn + 1 = __ arccos(__ xn), 4 2 with x0 = 0.4, find the values of x1, x2, x3 and x4. Give your answers to 4 decimal places. (3 marks) y = f(x) A C 1 D 2 x −1 −2 B 291 Chapter 10 d Using x0 = 1.7 as a first approximation to the root at D, apply the Newton–Raphson procedure once to f(x) to find a second approximation to the root, giving your answer to 3 decimal places. (4 marks) e By considering the change of sign of f(x) over an appropriate interval, show that the answer to part d is accurate to 3 decimal places. (2 marks) Challenge f(x) = x6 + x 3 − 7x2 − x + 3 The diagram shows a sketch of y = f(x). Points A and B are the points of inflection on the curve. y A O y = f(x) x B a Show that equation f0(x) = 0 can be written as: 7 − 15x i x = _________ √ _____ 7 − 3x iii x = _____ 15 15x + 3 b By choosing a suitable iterative formula and starting value, find an approximation for the x-coordinate of B, correct to 3 decimal places. 4 3 7 ii x = ________ 3 4 c Explain why you cannot use the same iterative formula to find an approximation for the x-coordinate of A. d Use the Newton–Raphson method to find an estimate for the x-coordinate of A, correct to 3 decimal places. Summary of key points 1 If the function f(x) is continuous on the interval [a, b] and f(a) and f(b) have opposite signs, then f(x) has at least one root, x, which satisfies a < x < b. 2 To solve an equation of the form f(x) = 0 by an iterative method, rearrange f(x) = 0 into the form x = g(x) and use the iterative formula xn + 1 = g(xn). 3 The Newton–Raphson formula for approximating the roots of a function f(x) is f(xn) xn + 1 = xn − _____ f9(xn) 292 11 Integration Objectives After completing this chapter you should be able to: ● Integrate standard mathematical functions including trigonometric and exponential functions and use the reverse of the chain rule to → pages 294–298 integrate functions of the form f(ax + b) ● Use trigonometric identities in integration → pages 298–300 ● Use the reverse of the chain rule to integrate more complex → pages 300–303 functions ● Integrate functions by making a substitution, using integration by → pages 303–313 parts and using partial fractions ● Use integration to find the area under a curve → pages 313–317 ● Use the trapezium rule to approximate the area under a curve. → pages 317–322 ● Solve simple differential equations and model real-life situations → pages 322–329 with differential equations Prior knowledge check 1 Differentiate: a (2x − 7) 6 b sin 5x x _ c e 3 2 ← Sections 9.1, 9.2, 9.3 _1 b find ∫4 f(x) dx 9 3 _1 Given f(x) = 8x 2 − 6x −2 a find ∫ f(x) dx ← Year 1, Chapter 13 3x + 22 Write ___________ as partial fractions. (4x − 1)(x + 3) ← Section 1.3 4 Find the area of the region R bounded by the curve y = x2 + 1, the x-axis and the lines x = −1 and x = 2. y 5 4 Integration can be used to solve differential equations. Archaeologists use differential equations to estimate the age of fossilised → Exercise 11K Q9 plants and animals. 3 2 1 −2 −1 O R 1 2 x ← Year 1, Chapter 13 293 Chapter 11 11.1 Integrating standard functions Integration is the inverse of differentiation. You can use your knowledge of derivatives to integrate familiar functions. Watch out This is true for all values of n 1 2 3 4 5 6 7 8 9 x ∫ xn dx = _____ +c n+1 ∫ ex dx = ex + c ∫ __x1 dx = ln|x| + c ∫ cos x dx = sin x + c ∫ sin x dx = −cos x + c ∫ sec2 x dx = tan x + c ∫ cosec x cot x dx = −cosec x + c ∫ cosec2 x dx = −cot x + c ∫ sec x tan x dx = sec x + c n+1 Example 1 dx it is usual to When finding ∫ __ x write the answer as ln |x| + c. The modulus sign removes difficulties that could arise when evaluating the integral for negative values of x. Notation Links dy For example, if y = cos x then ___ = −sin x. dx This means that ∫ (−sin x) dx = cos x + c and hence ∫ sin x dx = −cos x + c. b cos x − 2ex) dx ∫ (_____ sin2 x ∫ 2 cos x dx = 2 sin x + c ∫ __3x dx = 3 ln|x| + c ∫ √x dx = ∫ x __1 2 2 __32 dx = __ 3x + c 3 √__ dx So ∫ (2 cos x + __ x − x) 2 __32 = 2 sin x + 3 ln|x| − __ 3x + c cos x _____ cos x ____ 1 b _____ × = cot x cosec x = 2 sin x sin x sin x ∫ (cot x cosec x) dx = −cosec x ∫ 2ex dx = 2ex cos x So ∫ (_____ − 2ex dx) sin2 x = −cosec x − 2ex + c 294 ← Section 9.1 1 Find the following integrals. __ 3 a ∫ (2 cos x + __ x − √ x) dx a except −1. Integrate each term separately. 4. Use 3 . Use 1 . Use This is an indefinite integral so don't forget the +c. Look at the list of integrals of standard functions and express the integrand in terms of these standard functions. Remember the minus sign. Integration Example 2 Problem-solving Given that a is a positive constant and 3a 2x + 1 ∫a ______ x dx = ln 12, find the exact value of a. ∫a 3a ______ 2x + 1 x Integrate as normal and write the limits as a and 3a. Substitute these limits into your integral to get an expression in a and set this equal to ln12. Solve the resulting equation to find the value of a. dx (2 + x) dx = [2x + ln x] 3a a = (6a + ln 3a) − (2a + ln a) = ∫a 3a 1 __ Separate the terms by dividing by x, then integrate term by term. 3a = 4a + ln (___ a) = 4a + ln 3 So, 4a + ln 3 = ln 12 4a = ln 12 − ln 3 4a = ln 4 1 a = __ ln 4 4 Exercise Remember the limits are a and 3a. Substitute 3a and a into the integrated expression. a Use the laws of logarithms: ln a − ln b = ln (_) b 12 ln 12 − ln 3 = ln (___) = ln 4 3 Online Use your calculator to check your value of a using numerical integration. 11A 1 Integrate the following with respect to x. 5 2 a 3 sec2 x + __ + _2 x x c 2(sin x − cos x + x) 2 e 5ex + 4 cos x − _2 x 1 1 1 g __ + _2 + __3 x x x i 2 cosec x cot x − sec2 x 2 Find the following integrals. 1 1 a ∫ (_____ + __ dx cos2 x x2) 1 + cos x _____ 1+x c ∫ (_______ + 2) dx sin2x x e g i b 5ex − 4 sin x + 2x3 2 d 3 sec x tan x − __ x 1 f ___ + 2 cosec2 x 2x h ex + sin x + cos x 1 2 j ex + __ x − cosec x b d ∫ sin x(1 + sec2 x) dx ∫ cosec2 x(1 + tan2 x) dx ∫ sec2 x(1 + ex cos2 x) dx f h j sin x + 2ex) dx ∫ (______ cos2 x 1 1 + __ dx ∫ (_____ sin2x x) ∫ cos x(1 + cosec2 x) dx ∫ sec2 x(1 − cot2 x) dx ∫ 1 + sin x + cos2 x sec x) dx ∫ (_______ cos2 x 3 Evaluate the following. Give your answers as exact values. 7 61 + x b ∫ _____ dx a ∫3 2ex dx 1 x3 c ∫ π π _ 2 −5 sin x dx d ∫− 0 π _ 4 sec x(sec x + tan x) dx Watch out When applying limits to integrated trigonometric functions, always work in radians. 295 Chapter 11 E/P 2a 3x − 1 _1 4 Given that a is a positive constant and ∫a ______ x dx = 6 + ln (2), find the exact value of a. E/P 5 Given that a is a positive constant and ∫ E/P 6 Given ∫ (3ex + 6e−2x) dx = 0, find the value of b. E/P 4 1 _3 7 f(x) = _8 x 2 − __ x, x > 0 lna ln1 ex + e−x dx = __ 7 , find the exact value of a. 48 b (4 marks) (4 marks) (4 marks) 2 a Solve the equation f(x) = 0. (2 marks) b Find ∫ f(x) dx. c Evaluate ∫ f(x) dx, giving your answer in the form p + q ln r, where p, q and r are (2 marks) 4 1 rational numbers. (3 marks) 11.2 Integrating f(ax + b) If you know the integral of a function f(x) you can integrate a function of the form f(ax + b) using the reverse of the chain rule for differentiation. Example 3 Find the following integrals. a ∫ cos (2x + 3) dx b ∫ e4x + 1 dx a Consider y = sin (2x + 3): dy ___ = cos (2x + 3) × 2 dx 1 So ∫ cos (2x + 3) dx = __ 2 sin (2x + 3) + c b Consider y = e4x + 1: dy ___ = e4x + 1 × 4 dx 1 4x + 1 So ∫ e 4x + 1 dx = __ +c 4e c Consider y = tan 3x: dy ___ = sec2 3x × 3 dx 1 So ∫ sec2 3x dx = __ 3 tan 3x + c 296 c ∫ sec2 3x dx Integrating cos x gives sin x, so try sin (2x + 3). Use the chain rule. Remember to multiply by the derivative of 2x + 3 which is 2. This is 2 times the required expression so you need to divide sin (2x + 3) by 2. The integral of ex is ex, so try e4x + 1. This is 4 times the required expression so you divide by 4. Recall 6 . Let y = tan 3x and differentiate using the chain rule. This is 3 times the required expression so you divide by 3. Integration In general: ■ ∫ Watch out You cannot use this method to integrate an expression such as cos (2x2 + 3) since it is not in the form f(ax + b). 1 f′(ax + b) dx = __ f(ax + b) + c a Example 4 Find the following integrals: 1 a ∫ ______ dx 3x + 2 b ∫ (2x + 3)4 dx 1 Integrating __ x gives ln|x| so try ln (3x + 2). a Consider y = ln (3x + 2) dy 1 So ___ = ______ × 3 dx 3x + 2 1 1 So ∫ ______ dx = __ ln |3x + 2| + c 3x + 2 3 The 3 comes from the chain rule. It is 3 times the required expression, so divide by 3. b Consider y = (2x + 3)5 dy So ___ = 5 × (2x + 3)4 × 2 dx = 10 × (2x + 3)4 1 So ∫ (2x + 3)4 dx = ___ (2x + 3)5 + c 10 To integrate (ax + b)n try (ax + b)n + 1. The 5 comes from the exponent and the 2 comes from the chain rule. This answer is 10 times the required expression, so divide by 10. 11B Exercise 1 Integrate the following: a sin (2x + 1) b 3e2x c d cos (1 − 2x) 4ex + 5 e cosec2 3x g 3 sin (_1 2x + 1) Hint For part a consider y = cos(2x + 1). You do not need to write out this step once you are confident with using this method. f sec 4x tan 4x h sec2 (2 − x) i cosec 2x cot 2x 2 Find the following integrals. a c e ∫ (e2x − _21 sin (2x − 1)) dx ∫ sec2 2x(1 b d + sin 2x) dx ∫ (e3 − x + sin (3 − x) + cos (3 − x)) dx 3 Integrate the following: 1 a _____ 2x + 1 3 e _____ 1 − 4x j cos 3x − sin 3x ∫ (ex + 1)2 dx ∫ 1 3 − 2 cos _2 x _________ dx 1 2_ sin 2 x 1 b _______ 2 (2x + 1) c (2x + 1)2 3 d _____ 4x − 1 3 f _______ 2 (1 − 4x) g (3x + 2)5 3 h _______3 (1 − 2x) 297 Chapter 11 4 Find the following integrals. 4 a ∫(3 sin (2x + 1) + _____) dx 2x + 1 1 1 1 + _____ + _______2) dx c ∫(______ 2 1 + 2x sin 2x (1 + 2x) b ∫ (e5x + (1 − x)5) dx d 1 ∫((3x + 2)2 + _______ dx (3x + 2)2) c ∫ 5 Evaluate: a ∫ 3π __ π _ 4 4 cos (π − 2x) dx b ∫ 1 _______ 12 _1 2 dx (3 − 2x)4 5π __ 18 2π __ 9 sec2(π − 3x) dx d 5 ∫2 ________ dx 7 − 2x 3 E/P 6 Given ∫3 (2x − 6)2 dx = 36, find the value of b. (4 marks) E/P e8 1 1 7 Given ∫ 2 ___ dx = __, find the value of k. e kx 4 (4 marks) E/P b π __ 8 Given ∫_π_ (1 − π sin kx) dx = π(7 − 6√ 2), __ 3k 4k find the exact value of k. (7 marks) Problem-solving Calculate the value of the indefinite integral in terms of k and solve the resulting equation. Challenge 11 1 1 __ 41 Given ∫5 ______ dx = __ a ln(17), and that a and b ax + b are integers with 0 < a < 10, find two different pairs of values for a and b. 11.3 Using trigonometric identities ■ Trigonometric identities can be used to integrate expressions. This allows an expression that cannot be integrated to be replaced by an identical expression that can be integrated. Example Links Make sure you are familiar with the standard trigonometric identities. The list of identities in the summary of Chapter 7 will be useful. ← page 196 5 Find ∫ tan2 x dx Since sec2 x ; 1 + tan2 x tan2 x ; sec2 x − 1 So ∫ tan2 x dx = ∫ (sec2 x − 1) dx = ∫ sec2 x dx − ∫ 1 dx = tan x − x + c 298 You cannot integrate tan2 x but you can integrate sec2 x directly. Using 6. Integration Example 6 Show that ∫ π _ 8 π __ 12 sin2x dx = π ___ 48 __ + 1 − √2 _____ You cannot integrate sin2x directly. Use the trigonometric identity to write it in terms of cos 2x. 8 Recall cos 2x ≡ 1 − 2 sin 2x 1 So sin 2x ≡ _(1 − cos 2x) 2 __ π π 8 8 1 1 So ∫_π sin2x dx = ∫_π (_ − _ cos 2x) dx 12 12 2 2 Use the reverse chain rule. If y = sin 2x, dy ___ = 2 cos 2x. Adjust for the constant. dx Substitute the limits into the integrated expression. π __ 8 1 1 = [_x − __ sin 2x] π ___ 2 4 12 π π 1 = (− sin ( )) − (___ − __ sin (__)) 16 4 4 24 4 6 π ___ π ___ 1 ___ π π 1 1 = (___ − ( )) − (___ − (__)) 16 4 2 24 4 2 √2 1 _______ π π 1 1 √2 = (___ − ___) + (__ − ______) 16 24 4 2 2 1 − √2 3π 2π = (___ − ___) + _______ 48 48 8 __ 1 − √2 π = ___ + _______ 48 8 Watch out This is a 'show that' question so don't use your calculator to simplify the fractions. Show each line of your working carefully. 7 Find: a ∫ sin 3x cos 3x dx a You will save lots of time in your exam if you are familiar with the exact values for trigonometric functions given in radians. π Write sin (_) in its rationalised denominator form, __ 4 √2 1__ . This will make it easier to ___ rather than ___ 2 √2 simplify your fractions. __ Example Problem-solving b ∫ (sec x + tan x)2 dx ∫ sin 3x cos 3x dx = ∫ __21 sin 6x dx _1 1 = − __ 2 × 6 cos 6x + c 1 = − __ 12 cos 6x + c b (sec x + tan x)2 ≡ sec2x + 2 sec x tan x + tan2x ≡ sec2x + 2 sec x tan x + (sec2x − 1) ≡ 2 sec2x + 2 sec x tan x − 1 So ∫ (sec x + tan x)2 dx = ∫ (2 sec2x + 2 sec x tan x − 1) dx = 2 tan x + 2 sec x − x + c Remember sin 2A ≡ 2 sin A cos A, so sin 6x ≡ 2 sin 3x cos 3x. Use the reverse chain rule. 1 Simplify _21 × _16 to __ 12 Multiply out the bracket. Write tan2x as sec2x − 1. Then all the terms are standard integrals. Integrate each term using 6 and 9 . 299 Chapter 11 11C Exercise 1 Integrate the following: For part a, use 1 + cot2x ≡ cosec2x. For part c, use sin 2A ≡ 2 sin A cos A, making a suitable substitution for A. b cos2 x c sin 2x cos 2x d (1 + sin x)2 e tan2 3x f (cot x − cosec x)2 g (sin x + cos x)2 h sin2 x cos2 x 2 Find the following integrals. 1 − sin x dx a ∫ ________ cos2 x d g cos x dx ∫ _____ sin2 x 2 ∫ (cos x − sin x)2 dx 1 + cos x dx ∫ ________ sin2 x e dx ∫ ________ sin2 x (1 + cos x)2 ∫ (cos x − sec x)2 dx h _ 2 2+π 3 Show that ∫_π sin2 x dx = _____ 8 4 cos 2x dx ∫ _____ cos2 x f ∫ (cot x − tan x)2 dx cos 2x dx ∫ _________ 1 − cos2 2x c ∫0 π _ 4 (1 + sin x) ________ 2 cos2 x dx d ∫ π _ 2 3π __ 8 5 a By expanding sin (3x + 2x) and sin (3x − 2x) using the double-angle formulae, or otherwise, show that sin 5x + sin x ≡ 2 sin 3x cos 2x. sin 2x ________ 1 − sin2 2x dx (4 marks) (3 marks) 6 f(x) = 5 sin2 x + 7 cos2 x a Show that f(x) = cos 2x + 6. b Hence, find the exact value of E/P c i b Hence find ∫ sin 3x cos 2x dx E/P j (cos 2x − 1)2 (4 marks) 4 Find the exact value of each of the following: π π _ _ 1 3 4 2 dx b ∫ a ∫_π _________ π (sin x − cosec x) dx _ 2 2 6 sin x cos x 6 E/P 1 i _________ 2 sin x cos2 x b π E/P Hint a cot2 x ∫0 π _ 4 (3 marks) f(x) dx. (4 marks) 7 a Show that cos 4x ≡ _8 cos 4x + _2 cos 2x + _8 (4 marks) 1 1 3 b Hence find ∫ cos4 x dx. (4 marks) 11.4 Reverse chain rule f ′(x) If a function can be written in the form k___, you can integrate it using the reverse of the chain rule f(x) for differentiation. Example 8 Problem-solving Find a 2x dx ∫ _____ 2 x +1 300 b cos x ∫ ________ dx 3 + 2 sin x If f(x) = 3 + 2 sin x, then f′(x) = 2 cos x. By adjusting for the constant, the numerator is the derivative of the denominator. Integration a Let I= 2x ∫ _____ dx 2 x +1 Consider y = ln|x2 + 1| Then This is equal to the original integrand, so you don't need to adjust it. dy _____ 1 ___ = × 2x dx x2 + 1 I = ln|x2 + 1| + c So Since integration is the reverse of differentiation. cos x I = ∫ ________ dx 3 + 2 sin x b Let y = ln|3 + 2 sin x| Let dy ___ dx 1 = ________ × 2 cos x 3 + 2 sin x I= So _1 2 ln|3 + 2 sin x| + c ■ To integrate expressions of the form f ′(x) ∫ k ____ dx, try ln|f(x)| and differentiate f(x) to check, and then adjust any constant. Try differentiating y = ln|3 + 2 sin x|. The derivative of ln|3 + 2 sin x| is twice the original integrand, so you need to divide it by 2. Watch out You can't use this method to 1 integrate a function such as _____ because the x2 + 3 derivative of x2 + 3 is 2x, and the top of the fraction does not contain an x term. You can use a similar method with functions of the form kf'(x)(f(x))n. Example 9 Find: a ∫3 cos x sin2 x dx a Let b ∫x(x2 + 5)3 dx I = ∫ 3 cos x sin2 x dx Consider y = sin3 x dy ___ dx So b Let Then let = 3 sin2 x cos x This is equal to the original integrand, so you don't need to adjust it. I = sin3 x + c I = ∫x(x2 + 5)3 dx y = (x2 + 5)4 dy ___ dx = 4(x2 + 5)3 × 2x = 8x(x2 + 5)3 So Try differentiating sin3 x. 1 2 4 I = ___ 8 (x + 5) + c Try differentiating (x2 + 5)4. The 2x comes from differentiating x2 + 5. This is 8 times the required expression so you divide by 8. ■ To integrate an expression of the form ∫ k f ′(x)(f(x))n dx, try (f(x))n + 1 and differentiate to check, and then adjust any constant. 301 Chapter 11 Example 10 cosec2 x Use integration to find ∫ __________3 dx (2 + cot x) This is in the form ∫k f9(x)(f(x))n dx with f(x) = 2 + cot x and n = −3. cosec2 x Let I = ∫ __________3 dx (2 + cot x) Let y = (2 + cot x)−2 dy ___ dx Use the chain rule. = −2(2 + cot x)−3 × (−cosec2x) This is 2 times the required answer so you need to divide by 2. = 2(2 + cot x)−3 cosec2x 1 −2 + c So I = __ 2 (2 + cot x) Example 11 θ 15 π Given that ∫0 5 tan x sec4 x dx = ___ where 0 < θ < _, find the exact value of θ. 2 4 Let I = This is in the form ∫k f9(x)(f(x))n dx with f(x) = sec x and n = 4. ∫0 5 tan x sec4 x dx θ Let y = sec4 x dy ___ dx = 4 sec3 x × sec x tan x This is _45 times the required answer so you need to divide by _54 = 4 sec4 x tan x θ 5 15 So I = [__sec 4 x] = ___ 4 4 0 (4 5 __ 5 __ 4) − (4 5 __ sec 4 θ)= 4 sec 4 0 15 ___ 5 15 sec 4 θ − __ = ___ 4 4 5 __ 20 sec 4 θ = ___ 4 4 sec 4 θ = 4 302 Take the 4th root of both sides. Online Check your solution by using your calculator. 11D 1 Integrate the following functions. e2x x ______ a ______ b x2 + 4 e2x + 1 2x e cos 2x d ________ e ________ 2x 3 3 + sin 2x (e + 1) g xe x2 1 1 sec 0 = _____ = __ = 1 cos 0 1 π π 3π 5π 1__ are θ = − __, __, __, __ . . . The solutions to cos θ = ± ___ 4 4 4 4 √2 π The only solution within the given range for θ is __4 sec θ = ± √ 2 π θ = __ 4 Exercise Substitute the limits into the integrated expression. h cos 2x (1 + sin 2x)4 Hint x c ________ (x2 + 4)3 sin 2x f __________3 (3 + cos 2x) i sec2 x tan2 x Decide carefully whether each expression f9(x) is in the form k____ or f(x) kf9(x)(f(x))n . j sec2 x (1 + tan2 x) Integration 2 Find the following integrals. a c e ∫(x + 1)(x2 + 2x + 3)4 dx ∫ sin5 3x cos 3x dx e2x dx ∫ _______ e2x + 3 b d f ∫ cosec2 2x cot 2x dx ∫ cos x esin x dx ∫ x(x2 + 1) dx _3 2 g ∫ (2x + 1)√x2 + x + 5 dx h 2x + 1 _________ dx ∫ _________ √ x2 + x + 5 i sin x cos x _________ dx ∫ _________ √ j sin x cos x ∫ ________ dx cos 2x + 3 b ∫ ________ cos 2x + 3 3 Find the exact value of each of the following: a c E/P P E/P ∫0 (3x2 + 10x)√x 3 + 5x 2 + 9 dx ________ 3 x dx ∫4 ______ 2 x −1 7 d 4 Given that ∫0 kx2e x 3 dx = _3 (e8 − 1), find the value of k. k 2π __ π _ 9 9 π _6 sin 3x _______ 1 − cos 3x dx ∫0 sec2x e4 tan x dx 4 2 (3 marks) 5 Given that ∫0 4 sin 2x cos4 2x dx = _5 where 0 < θ < π, find the exact value of θ. θ 4 cos x 6 a By writing cot x = _____, find ∫ cot x dx. sin x b Show that ∫tan x dx ≡ ln|sec x| + c. (2 marks) (3 marks) 11.5 Integration by substitution ■ Sometimes you can simplify an integral by changing the variable. The process is similar to using the chain rule in differentiation and is called integration by substitution. In your exam you will often be told which substitution to use. Example 12 Find ∫x√ 2x + 5 dx using the substitutions: ______ a u = 2x + 5 a Let b u2 = 2x + 5 I = ∫ x√ 2x + 5 dx ______ Let u = 2x + 5 du So ___ = 2 dx 1 So dx can be replaced by __ 2 du. ______ √ 2x __ You need to replace each 'x' term with a corresponding 'u' term. Start by finding the relationship between dx and du. So dx = _12 du. Next rewrite the function in terms of u = 2x + 5. _1 + 5 = √u = u 2 u−5 x = _____ 2 Rearrange u = 2x + 5 to get 2x = u − 5 and hence u−5 x = _____ 2 303 Chapter 11 u − 5 _1 1 So I = ∫ (_____)u 2 × __ 2 du 2 Rewrite I in terms of u and simplify. _1 1 2 = ∫ __ 4 (u − 5)u du _1 1 __32 2 = ∫ __ 4 (u − 5u) du Multiply out the brackets and integrate using rules from your Year 1 book. ← Year 1, Chapter 13 __3 5 __ 5u 2 u 2 ________ −+c = × _____ 5 ___ 3 4 4 × __ 2 2 1 ____ 3 ___ 3 = u2 ___ 10 − 5u 2 ______ 6 Simplify. +c 5 ___ 3 (2x + 5)2 5(2x + 5)2 So I = __________ − _________ + c 6 10 b Let Finally rewrite the answer in terms of x. I = ∫ x√ 2x + 5 dx ______ First find the relationship between dx and du. u2 = 2x + 5 Using implicit differentiation, cancel 2 and rearrange to get dx = udu. du 2u ___ = 2 dx So replace dx with udu. ______ √ 2x +5 =u Rewrite the integrand in terms of u. You will need to make x the subject of u2 = 2x + 5. and u2 − 5 x = _____ 2 So u2 − 5 I = ∫ (_____) u × udu 2 u4 5u2 = ∫ (__ − ___) du 2 2 Multiply out the brackets and integrate. u5 5u3 = ___ − ____ + c 10 6 5 __ (2x + 5) ________ 2 So I= 10 __ 3 5(2x + 5) ____________ 2 − 6 +c Rewrite answer in terms of x. Example 13 Use the substitution u = sin x + 1 to find ∫ cos x sin x (1 + sin x)3 dx Let Let I = ∫ cos x sin x (1 + sin x)3 dx u = sin x + 1 du ___ = cos x dx So substitute cos x dx with du. 304 First replace the dx. cos x appears in the integrand, so you can write this as du = cos x dx and substitute. Integration Use u = sin x + 1 to substitute for the remaining terms, rearranging where required to get sin x = u − 1. (sin x + 1)3 = u3 sin x = u − 1 I = ∫ (u − 1)u3 du So Rewrite I in terms of u. = ∫ (u4 − u3) du u5 u4 = ___ − ___ + c 5 4 Multiply out the brackets and integrate in the usual way. (sin x + 1)5 (sin x + 1)4 I = _________ − _________ + c 5 4 So Example 14 Problem-solving Although it looks different, ∫ sin 2x(1 + sin x)3 dx can be integrated in exactly the same way. Remember sin 2x ≡ 2 sin x cos x, so the above integral would just need adjusting by a factor of 2. 1 ______ Prove that ∫ ________ dx = arcsin x + c. √ 1 − x2 Let 1 _____ dx I = ∫ ______ √ 1 − x2 This substitution is not obvious at first. Think how the integral will be transformed by using trigonometric identities. ← Chapter 7 Let x = sin θ dx ___ = cos θ dθ dx The substitution is in the form x = f(θ), so find ___ dθ to work out the relationship between dx and dθ. So replace dx with cosθ dθ. 1 ________ cosθ dθ I = ∫ ________ √ 1 − sin2θ 1 ______ = ∫ _______ cosθ dθ √ cos2θ 1 = ∫ _____ cos θ dθ cos θ Make the substitution, and replace dx with cosθ dθ. Remember sin2 A + cos2 A ≡ 1 = ∫1dθ = θ + c x = sin θ = θ = arcsin x So Remember to use your substitution to write the final answer in terms of x, not θ. I = arcsin x + c Example 15 Use integration by substitution to evaluate: a ∫0 x(x + 2 a Let Let 1)3 dx I= b ∫0 ∫0 x(x + 1)3 dx 2 π _ 2 ________ cos x√ 1 + sin x dx Watch out If you use integration by substitution to evaluate a definite integral, you have to be careful of whether your limits are x values or u values. You can use a table to keep track. u=x+1 du ___ dx =1 305 Chapter 11 so replace dx with du and replace (x + 1)3 with u3, and x with u − 1. x 2 0 u 3 1 Change the limits. When x = 2, u = 2 + 1 = 3 and when x = 0, u = 1. ∫1 (u − 1)u3 du 3 = ∫1 (u4 − u3) du 3 So I = Note that the new u limits replace their corresponding x limits. u5 u4 = [___ − ___] 5 4 1 3 Multiply out and integrate. Remember there is no need for a constant of integration with definite integrals. 243 81 1 1 = (_____ − ___) − (__ − _) 5 4 5 4 = 48.4 − 20 = 28.4 b The integral can now be evaluated using the limits for u without having to change back into x. ∫0 cos x√1 + sin x dx __ π _______ 2 du u = 1 + sin x ⇒ ___ = cos x, so replace dx _______ cos x dx with du and replace √ 1 + sin x with _1 u 2. x u 2 0 2 π __ 1 So I = ∫1 2 _1 u 2 du Use u = 1 + sin x. Remember that limits for integrals involving trigonometric functions will always be in radians. π x = __, means u = 1 + 1 = 2 and x = 0, means 2 u = 1 + 0 = 1. Rewrite the integral in terms of u. 2 __ 2 __32 [3u]1 _3 __ 2 2 2 = (__ 32) − (3) __ 2 √ So I = __ 3 (2 2 − 1) c e You could also write convert the integral back into a function of x and use the original limits. 11E ∫x√1 + x dx; u = 1 + x _____ b ∫sin3 x dx; u = cosx d ∫sec2 x tan x√1 + tan x dx; u2 = 1 + tan x ________ f 2 Use the substitutions given to find the exact values of: a c d 306 ∫0 x√x + 4 dx; u = x + 4 5 π _____ ∫0 sin x√3 cos x + 1 dx; u = cos x 2 π _ 3 __ Problem-solving 1 Use the substitutions given to find: a __ Remember that 22 = √ 8 = 2√ 2 . __ 3 Exercise Replace each term in x with a term in u in the usual way. ________ ∫0 sec x tan x√sec x + 2 dx; u = sec x _______ b Hint 1 + sin x ∫_______ cos x dx; u = sin x 2 __ dx; u = √ x ∫ ________ √ x (x − 4) ____ ∫sec4 x dx; u = tan x ∫0 x(2 + x)3 dx; u = 2 + x 2 First apply a trigonometric identity. e 1 __ dx; u = √ x ∫1 ________ √ __ 4 x (4x − 1) Integration P P 3 By choosing a suitable substitution, find: x _____ dx a ∫x(3 + 2x)5 dx b ∫ ________ √1 + x 4 E 5 E/P 6 c By choosing a suitable substitution, find the exact values of: ____ 7 5 1______ dx c a ∫ x√ 2 + x dx b ∫ ________ 2 2 1 + √x − 1 x dx π _ sin 2θ ∫0 _______ dθ 1 + cos θ 2 Using the substitution u2 = 4x + 1, or otherwise, find the exact value of ∫6 20 ______ 8x _____ dx. √ 4x +1 ln4 e4x a Use the substitution u2 = ex − 2 to show that ∫ _____ dx = __ + c ln d, where a, b, c ln3 ex − 2 b and d are integers to be found. E/P 7 1 ______ Prove that −∫ _______ dx = arccos x + c. √ 1 − x2 E/P 8 Use the substitution u = cos x to show π _ ∫0 3 sin3x cos2x dx = 47 ___ 480 (7 marks) Hint Use exact trigonometric values to change the limits in x to limits in u. (7 marks) Using a suitable trigonometric substitution for x, find ∫_1 x2√ 1 − x 2 dx. √3 __ 9 (8 marks) (5 marks) __ E/P ∫ ______ √ x2 + 4 _______ 2 _______ 2 (8 marks) Challenge By using a substitution of the form x = k sin u, show that ∫ _______ 2 √9 − x 1 ________ dx = − _______ + c 2 2 9x x √9 − x 11.6 Integration by parts You can rearrange the product rule for differentiation: dv du d ___ (uv) = u ___ + v ___ dx dx dx dv du d u ___ = ___ (uv) − v ___ dx dx dx dv du d ∫u ___ dx = ∫ ___ (uv) dx − ∫v ___ dx dx dx Links u and v are both functions of x. ← Section 9.4 Differentiating a function and then integrating it leaves the original function unchanged. d So, ∫ ___ (uv) dx = uv. dx du dv ■ This method is called integration by parts. ∫ u___ dx = uv − ∫v ___ dx dx dv To use integration by parts you need to write the function you are integrating in the form u ___ dx dv You will have to choose what to set as u and what to set as ___ dx 307 Chapter 11 Example 16 Find ∫x cos x dx I = ∫ x cos x dx du u=x ⇒ ___ = 1 dx dv ___ = cos x ⇒ v = sin x dx Using the integration by parts formula: Let I = x sin x − ∫ sin x × 1 dx = x sin x + cos x + c Example 17 Problem-solving For expressions like x cos x, x2sin x and x 3ex let u equal the xn term. When the expression involves lnx, for example x2 ln x, let u equal the ln x term. dv Let u = x and ___ = cos x. dx du dv Find expressions for u, v, ___ and ___ dx dx dv Take care to differentiate u but integrate ___ dx du Notice that ∫v ___ dx is a simpler integral than dx dv ∫u ___ dx. dx Find ∫x2 ln x dx Let I = ∫x2 ln x dx du 1 u = ln x ⇒ ___ = __ dx x dv x3 ___ = x2 ⇒ v = ___ dx 3 3 3 x x 1 I = ___ ln x − ∫ ___ × __ x dx 3 3 x3 x2 ___ ln x − ∫ ___ dx 3 3 x3 x3 ___ +c = ln x − 9 3 dv Since there is a ln x term, let u = ln x and ___ = x2. dx du dv Find expressions for u, v, ___ and ___ dx dx dv Take care to differentiate u but integrate ___ dx Apply the integration by parts formula. du Simplify the v ___ term. dx It is sometimes necessary to use integration by parts twice, as shown in the following example. Example 18 Find ∫x2ex dx Let So I = ∫ x2ex dx du u = x2 ⇒ ___ = 2x dx dv ___ = ex ⇒ v = ex dx I = x2ex − ∫2xexdx du u = 2x ⇒ ___ = 2 dx dv ___ = e x ⇒ v = ex dx 308 dv There is no ln x term, so let u = x2 and ___ = ex . dx du dv Find expressions for u, v, ___ and ___ dx dx dv Take care to differentiate u but integrate ___ dx Apply the integration by parts formula. Notice that this integral is simpler than I but still not one you can write down. It has a similar structure to I and so you can use integration by dv parts again with u = 2x and ___ = ex. dx Integration So I = x2ex − (2xex − ∫ 2exdx) Apply the integration by parts formula for a second time. = x2ex − 2xex + ∫ 2exdx = x2ex − 2xex + 2ex + c Example 19 Evaluate ∫ ln x dx, leaving your answer in terms of natural logarithms. 2 1 Let I = ∫1 2 ln x dx = ∫1 2 Write the expression to be integrated as ln x × 1, dv then u = ln x and ___ = 1. dx Remember if an expression involves ln x you should always set u = ln x. ln x × 1 dx du 1 u = ln x ⇒ ___ = __ dx x dv ___ = 1 ⇒v = x dx 2 I = [x ln x] 1 − ∫1 2 1 x × __ x dx = (2 ln 2) − (1 ln 1) − = 2ln 2 − [x] 21 ∫1 2 Problem-solving du Apply limits to the uv term and the ∫ v ___ dx term dx separately. 1 dx = 2 ln 2 − (2 − 1) Evaluate the limits on uv and remember ln 1 = 0. = 2 ln 2 − 1 Exercise 11F 1 Find the following integrals. a d ∫x sin x dx ∫x sec x tan x dx b e ∫xex dx x dx ∫ _____ sin2 x c Hint You will need to use these standard results. In your exam they will be given in the formulae booklet: ∫x sec2 x dx • 2 Find the following integrals. a d ∫3 ln x dx ∫(ln x)2 dx b e ln x ∫x lnx dx c ∫ ___ dx x3 ∫(x2 + 1) ln x dx 3 Find the following integrals. a ∫x2e−x dx b ∫x2 cos x dx 4 Evaluate the following: a e ∫0ln2 xe2x dx 1 ∫0 4x(1 + x)3 dx • b ∫ π _ c • ∫12x2(3 + 2x)5 dx ∫0 x sin x dx π ∫0 x cos _14 x dx 2 • c g d π _ ∫ tan x dx = ln|sec x| + c ∫ sec x dx = ln|sec x + tan x| + c ∫ cot x dx = ln|sin x| + c ∫ cosec x dx = −ln|cosec x + cot x| + c ∫2x2 sin 2x dx ∫0 x cos x dx ∫0 sin x ln (sec x) dx 2 π _ e ∫2x2 sec2x tan x dx d ∫1 2 ln x ___ x2 dx 3 309 Chapter 11 E E/P 5 a Use integration by parts to find ∫x cos 4x dx. (3 marks) b Use your answer to part a to find ∫x2 sin 4x dx. (3 marks) 6 a Find ∫√ 8 − x dx. _____ (2 marks) b Using integration by parts, or otherwise, show that ∫(x − 2) √8 − x dx = − __25(8 − x) (x + 2) + c _____ _3 (6 marks) 2 c Hence find ∫ (x − 2)√ 8 − x dx. ____ 7 (2 marks) 4 E/P 7 a Find ∫sec2 3x dx. (3 marks) b Using integration by parts, or otherwise, find ∫x sec2 3x dx. π _ c Hence show that ∫_π x sec2 3x dx = pπ − q ln 3, finding the exact values of the 18 constants p and q. (6 marks) 9 (4 marks) 11.7 Partial fractions ■ Partial fractions can be used to integrate algebraic fractions. Using partial fractions enables an expression that looks hard to integrate to be transformed into two or more expressions that are easier to integrate. Links Make sure you are confident expressing algebraic fractions as partial fractions ← Chapter 1 Example 20 Use partial fractions to find the following integrals. 8x2 − 19x + 1 x−5 a ∫ ___________ dx b ∫ _______________2 dx (x + 1)(x − 2) (2x + 1)(x − 2) x−5 A B a ___________ ≡ _____ + ______ (x + 1)(x − 2) x + 1 x − 2 c 2 ∫ _____ dx 1 − x2 Split the expression to be integrated into partial fractions. So x − 5 ≡ A(x − 2) + B(x + 1) Let x = −1: −6 = A(−3) so A = 2 Let x = 2: −3 = B(3) so B = −1 x−5 So ∫ ___________ dx (x + 1)(x − 2) 2 1 = ∫ (_____ − ______)dx x+1 x−2 = 2 ln|x + 1| − ln|x − 2| + c | | (x + 1)2 = ln ______ + c x−2 Let x = −1 and 2. Rewrite the integral and integrate each term as in ← Section 11.2 Remember to use the modulus when using ln in integration. The answer could be left in this form, but sometimes you may be asked to combine the ln terms using the rules of logarithms. ← Year 1, Chapter 14 310 Integration b Let I = 8x − 19x + 1 ∫ _______________ dx (2x + 1)(x − 2)2 2 8x2 − 19x + 1 _______________ (2x + 1)(x − C A B ≡ _______ + ________2 + _____ 2x + 1 (x − 2) x−2 2)2 It is sometimes useful to label the integral as I. Remember the partial fraction form for a repeated factor in the denominator. 8x2 − 19x + 1 ≡ A(x − 2)2 + B(2x + 1) + C(2x + 1)(x − 2) Let x = 2: −5 = 0 + 5B + 0 so B = −1 25 __1 ___ 1 Let x = − _2 : 12 2 = 4 A + 0 + 0 so A = 2 Let x = 0: Then 1 = 4A + B − 2C 1 = 8 − 1 − 2C so C = 3 3 I = ∫(+ ______ dx − 2x + 1 (x − 2)2 x − 2) 1 _____ 2 = __ 2 ln|2x + 1| + x − 2 + 3 ln|x − 2| + c 1 = ln|2x + 1| + _____ + ln|x − 2|3 + c x−2 1 = ln|(2x + 1)(x − 2)3| + _____ + c x−2 2 c Let I = ∫ _____2 dx 1−x 2 2 A B _____ _____ _____ = = + ______ 1 − x2 (1 − x)(1 + x) 1 − x 1 + x So 2 _____ 1 ________ Rewrite the intergral using the partial fractions. Note that using I saves copying the question again. Don't forget to divide by 2 when integrating 1 ______ and remember that the integral of 2x + 1 1 ______ does not involve ln. (x − 2)2 Simplify using the laws of logarithms. Remember that 1 − x2 can be factorised using the difference of two squares. 2 = A(1 + x) + B(1 − x) Let x = −1 then 2 = 2B so B = 1 Let x = 1 then 2 = 2A so A = 1 1 1 So I = ∫(____ + _____) dx 1+x 1−x = ln|1 + x| − ln|1 − x| + c |1 − x| + c = ln 1+x ____ Rewrite the integral using the partial fractions. Notice the minus sign that comes from 1 integrating _____ 1−x When the degree of the polynomial in the numerator is greater than or equal to the degree of the denominator, it is necessary to first divide the numerator by the denominator. Example 21 9x2 − 3x + 2 dx Find ∫ _________ 9x2 − 4 Let 9x2 − 3x + 2 I = ∫ __________ dx 9x2 − 4 9x2 1 _____ 2) − 4 9x − 3x + 2 9x2 −4 First divide the numerator by 9x2 − 4. 9x2 ÷ 9x2 gives 1, so put this on top and subtract 1 × (9x2 − 4). This leaves a remainder of −3x + 6. −3x + 6 311 Chapter 11 so 6 − 3x I = ∫ (1 + _________ dx 9x2 − 4) 6 − 3x B A _________ ≡ ______ + ______ 2 3x − 2 3x + 2 9x − 4 2 Let x = __ 3 then 8 = −4B so B = −2 Factorise 9x2 − 4 and then split into partial fractions. 2 Let x = __ 3 then 4 = 4A so A = 1 So 2 1 I = ∫(1 + ______ − ______) dx 3x − 2 3x + 2 __ 2 1 = x + __ 3 ln|3x − 2| − 3 ln|3x + 2|+ c | | 3x − 2 _______ 1 = x + __ +c 3 ln (3x + 2)2 Exercise Rewrite the integral using the partial fractions. Integrate and don't forget the _13 Simplify using the laws of logarithms. 11G 1 Use partial fractions to integrate the following: 3x + 5 3x − 1 a ______________ b ______________ (x + 1)(x + 2) (2x + 1)(x − 2) 2 Find the following integrals. 2(x2 + 3x − 1) x3 + 2x2 + 2 a ∫ ______________ dx b ∫ ____________ dx (x + 1)(2x − 1) x(x + 1) E/P 2x − 6 c _________ (x + 3)(x − 1) c dx ∫ _________ 2 x −4 2 3 d ___________ (2 + x)(1 − x) d x +x2 ∫ __________ dx 3 − 2x − x2 2 4 1 3 f(x) = _____________ , x ≠ ± _ 2 (2x + 1)(1 − 2x) B A a Given that f(x) = ______ + ______ , find the value of the constants A and B. 2x + 1 1 − 2x b Hence find ∫f(x) dx, writing your answer as a single logarithm. c Find ∫ f(x) dx, giving your answer in the form ln k where k is a rational constant. 2 1 E/P 17 − 5x 3 4 f(x) = _____________2 , − __ < x < 2. (3 + 2x)(2 − x) 2 a Express f(x) in partial fractions. b Hence find the exact value of 17 − 5x dx, writing your answer in the form ∫0 ______________ (3 + 2x)(2 − x) 2 9x 2 + 4 2 5 f(x) = _______ , x ≠ ± __ 3 9x 2 − 4 (4 marks) (2 marks) (4 marks) 1 a + ln b, where a and b are constants to be found. E/P (3 marks) (5 marks) C B a Given that f(x) = A + _____ + _____ , find the values of the constants A, B and C. (4 marks) 3x − 2 3x + 2 b Hence find the exact value of Problem-solving _1 9x2 + 4 3 ______ Simplify the

integral as much as possible before ∫−_13 9x2 − 4 dx, writing your answer in the substituting your limits. form a + b ln c, where a, b and c are rational numbers to be found. (5 marks)
answer in the form a + ln b, ∫2 __________ x 3 + 2x 2 46 x2 (5 marks) where a and b are rational numbers to be found. E/P C 32x 2 + 4 B 7 ______________ ≡ A + ______ + ______ 4x + 1
form 1 (4x + 1)(4x − 1) 2 + k ln m , giving the values of the rational constants k and m. (5 marks) 11.8 Finding areas You need to be able to use the integration techniques from this
the lines x = 0 and x = 4, as shown in the diagram. Use integration to find the area of R. 9 2 _ 1 1__ ___ = = x −2 Remember ___ Area = ∫ x _1 x2 4 x O _1 2 = 9∫0 (4 + 3x) dx −2 = 6[(4
9 4 + 3x 1 √ 4 ______________ 9 ______ dx 0 √ 4 + 3x y= __1 Substitute the limits. __ = 6(√ 16 − √ 4) You don't need to give units when finding areas under graphs in pure maths. = 12 ■
use this formula if the two curves do not intersect between a and b. x 313 Chapter 11 y Example 23 The diagram shows part of the curves y = f(x) and y = g(x), π where f(x) = sin 2x
dx The region R is bounded by two curves. ∫0 (sin 2x − sin x cos 2x) dx Substitute the limits and functions given in the question. b __ π 2 __ π __1 1 3 2 = [− __ 2 cos 2x + 3 cos x] 0 =
technology. Exercise 11H 1 Find the area of the finite region R bounded by the curve with equation y = f(x), the x-axis and the lines x = a and x = b. π 2 c f(x) = ln x; a = 1, b = 2 b f(x)
region bounded by the curve y = f(x), the x-axis and the lines x = a and x = b where: x 4x − 1 b f(x) = _______2 ; a = 0, b = 2 a f(x) = _____________; a = 0, b = 2 (x + 2)(2x + 1) (x + 1)
equation, y = f(x), 4x + 3 1 where f(x) = _____________, x > _2 (x + 2)(2x − 1) Find the area of the shaded region bounded by the curve, the x-axis and the lines x = 1 and x = 2. (7 marks
shaded region bounded by the curve, the x-axis and the lines x = 2 and x = 4. y y = f(x) O P 2 4 x 5 The diagram shows a sketch of the curve with equation y = g(x), where g(x) = x sin
separately and then add the answers. Remember areas cannot be negative, so take the absolute value of any negative area. 6 The diagram shows a sketch of the curve with equation y
the exact area of the shaded region, giving your 2 answer in the form __ (a ln 2 + b) , where a and b are integers. 3 (5 marks) O 2 x 315 Chapter 11 E/P ________ 7 The diagram shows a
suitable substitution to find ∫ 3 cos x√ sin x + 1 dx. ________ (5 marks) c Show that the regions R1 and R2 have the same area, and find the exact value of this __ √ (3 marks) area in the
coordinates of any points of intersection of the two curves. b Find the area of the finite region bounded by the two curves. E/P 9 The diagram shows a sketch of part of the curves with
Find the area of region R1 in the form a√ 3 + ___ c , where a, b and c are integers to be found. (4 marks) __ __ c Show that the ratio of R2 : R1 can be expressed as (3√ 3 + 2π) : (3√ 3 −
first diagram is equal to the total shaded area in the second diagram, and state the exact value of this shaded area. y 1 O –1 316 y y = sin θ 1 2π θ O –1 y = sin 2θ 2π θ Integration P 11
R1 ii R2 iii R3 __ c Show that the ratio of areas R1 : R2 can be written as √ 2 : 2. Challenge π The diagram shows the curves y = sin 2x and y = cos x, 0 < x < __ 4 y y = sin 2x 1 y = cos
method to approximate the area beneath a curve. Consider the curve y = f(x): y O y = f(x) a To approximate the area given by ∫a y dx, you can divide b b x y the area up into n equal str
one of the strips. So you find y for x = a, x = a + h, x = a + 2h, x = a + 3h and so on up to x = b. You can label these values y0, y1, y2, y3, ... , yn. Hint Notice that for n strips there will
sum of the areas of these n trapeziums. You may recall from GCSE maths that the area of a trapezium like this: y1 y0 h is given by _12 (y0 + y1)h. The required area under the curve is
+ yn − 1 + yn) b ∫a y dx ≈ _12 h(y0 + 2(y1 + y2 ... + yn − 1) + yn) b or This formula is given in the formula booklet but you will need to know how to use it. ■ The trapezium rule: ∫a
region R π is bounded by the curve, the x-axis, the y-axis and the line x =__ 3 The table shows the corresponding values of x and y for y = sec x. x 0 y 1 π ___ 12 1.035 π __ 6 π __ 4 y π
b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places. c Explain with a reason whethe
__ 12 6 1.035 1.155 π __ 4 1.414 π Substitute h = ___ and the five y-values into 12 the formula. π __ 3 2 ∫a y dx ≈ __21 h(y0 + 2(y1 + y2 ... + yn − 1) + yn) Online b Explore under- and

cos 2x + 2, 0 < x < ___ 2 Find the exact value of the total shaded area on the diagram. y y = sin x + 2 y = cos 2x + 2 ← Section 12.3 E/p 39 Given that −6i + 40j + 16k = 3pi + (8 + qr)j + 2prk find the values of p, q and r. (3) ← Section 12.3 O x ← Section 11.8 357 Exam-style practice Mathematics A Level Paper 1: Pure Mathematics Time: 2 hours You must have: Mathematical Formulae and Statistical Tables, Calculator 1 2 3 4 π A curve C has parametric equations x = sin2 t, y = 2 tan t, 0 < t , __ 2 dy Find ___ in terms of t. dx Find the set of values of x for which a 2(7x − 5) − 6x < 10x − 7 (2) b |2x + 5| − 3 > 0 (4) c both 2(7x − 5) − 6x < 10x − 7 and |2x + 5| − 3 > 0. (1) The line with equation 2x + y − 3 = 0 does not intersect the circle with equation x2 + kx + y2 + 4y = 4 a Show that 5x 2 + (k − 20)x + 17 > 0. (4) b Find the range of possible values of k. Write your answer in exact form. (3) Prove, for an angle θ measured in radians, that the derivative of cos θ is −sin θ. You may assume the compound angle formula for cos (A ± B), and that sin h cos h − 1 ____ ________ = 1 and lim lim) = 0. h→0 (h) h→0 (h 5 (4) (5) f(x) = (3 + px) 6, x ∈ ℝ Given that the coefficient of x2 is 19 440, a find two possible values of p. (4) Given further that the coefficient of x is negative, 5 b find the coefficient of x5. 358358 (2) Exam-style practice 6 The point R with x-coordinate 2 lies on the curve with equation y = x2 + 4x − 2. The normal to the curve at R intersects the curve again at a point T. Find the coordinates of T, giving your answers in their simplest form. (6) 7 A geometric series has first term a and common ratio r. The second term of the series is 96 and the sum to infinity of the series is 600. (4) a Show that 25r2 − 25r + 4 = 0. b Find the two possible values of r. (2) For the larger value of r: c find the corresponding value of a. (1) (3) d find the smallest value of n for which S n exceeds 599.9. 8 The diagram shows the graph of f(x). The points B and D are stationary points of the graph. y D (0, 6) A C E (–10, 0) (–3, 0) O (4, 0) x B (–6, –11) Sketch, on separate diagrams, the graphs of: a y = | f(x)| b y = −f(x) + 5 c y = 2f(x − 3) 9 (3) (3) (3) Find all the solutions, in the interval 0 < x < 2π, to the equation 31 − 25 cos x = 19 − 12 sin 2 x, giving each solution to 2 decimal places. (5) 10 The value, V, of a car decreases over time, t, measured in years. The rate of decrease in value of the car is proportional to the value of the car at that time. (4) a Given that the initial value of the car is V 0, show that V = V 0e −kt The value of the car after 2 years is £25 000 and after 5 years is £15 000. (3) b Find the exact value of k and the value of V0 to the nearest hundred pounds. c Find the age of the car when its value is £5000. (3) 359 Exam-style practice 11 The diagram shows the positions of 4 cities: A, B, C and D. The distances, in miles, between each pair of cities, as measured in a straight-line, are labelled on the diagram. A new road is to be built between cities B and D. A 8 B 11 D 21 19 14 C a What is the minimum possible length of this road? Give your answer to 1 decimal place. b Explain why your answer to part a is a minimum. (7) (1) 12 A footballer takes a free-kick. The path of the ball towards the goal can be modelled by the equation y = −0.01x 2 + 0.22x + 1.58, x > 0, where x is the horizontal distance from the goal in metres and y is the height of the ball in metres. The goal is 2.44 m high. (3) a Rewrite y in the form A − B(x + C) 2 , where A, B and C are constants to be found. b Using your answer to part a, state the distance from goal at which the ball is at the greatest height and its height at this point. (2) c How far from goal is the football when it is kicked? (2) d The football is headed towards the goal. The keeper can save any ball that crosses the goal line at a height of up to 1.5 m. Explain with a reason whether the free kick will result in a goal. (2) 13 A box in the shape of a rectangular prism has a lid that overlaps the box by 3 cm, as shown. The length of the box is double the width, x cm. The height of the box is h cm. The box and lid can be created exactly from a piece of cardboard of area 5356 cm2. The box has volume, V cm3. 3 cm Height Width Length a Show that V = _3 (2678x − 9x 2 − 2x 3) Given that x can vary b use differentiation to find the positive value of x, to 2 decimal places, for which V is stationary. c Prove that this value of x gives a maximum value of V. d Find this maximum value of V. Given that V takes its maximum value, e determine the percentage of the area of cardboard that is used in the lid. 2 360 (5) (4) (2) (1) (2) Exam-style practice Mathematics A Level Paper 2: Pure Mathematics Time: 2 hours You must have: Mathematical Formulae and Statistical Tables, Calculator 1 The graph of y = ax2 + bx + c has a maximum at (−2, 8) and passes through (−4, 4). Find the values of a, b and c. (3) 2 The points P(6, 4) and Q(0, 28) lie on the straight line l1 as shown. y Q l2 P R O x l1 a Work out an equation for the straight line l1. (2) The straight line l2 is perpendicular to l1 and passes through the point P. b Work out an equation for the straight line l2. (2) c Work out the coordinates of R. (2) d Work out the area of △PQR. (3) 3 The function f is defined by f : x → e3x − 1, Find f−1(x) and state its domain. 4 A student is asked to solve the equation log4(x + 3) + log4(x + 4) = _2 The student's attempt is shown. x ∈ ℝ. (4) 1 1 log 4(x + 3) + log 4(x + 4) = __ 2 (x + 3) + (x + 4) = 2x + 7 = 2x = x= 2 2 −5 5 − __ 2 361361 Exam-style practice a Identify the error made by the student. b Solve the equation correctly. (1) (4) 5 The function p has domain −14 < x < 10, and is linear from (−14, 18) to (−6, −6) and from (−6, −6) to (10, 2). a Sketch y = p(x). (2) b Write down the range of p(x). (1) c Find the values of a, such that p(a) = −3. (2) 6 f(x) = x3 − kx2 − 10x + k a Given that (x + 2) is a factor of f(x), find the value of k. (2) b Hence, or otherwise, find all the solutions to the equation f(x) = 0, leaving your answers in __ (4) the form p ± √ q when necessary. 7 In △DEF, DE = x − 3 cm, DF = x − 10 cm and ∠EDF = 30°. Given that the area of the triangle is 11 cm2, (3) a show that x satisfies the equation x2 − 13x − 14 = 0 b calculate the value of x. (2) 8 9 3π π The curve C has parametric equations x = 6 sin t + 5 , y = 6 cos t − 2, − __ < t < ___ 3 4 2 2 a Show that the Cartesian equation of C can be written as (x + h) + (y + k) = c , where h, k and c are integers to be determined. (4) b Find the length of C. Write your answer in the form pπ, where p is a rational number to be found. (3) 4x + 7x ___________ 2 (x − 2)(x + 4) C B ≡ A + _____ + _____ x−2 x+4 a Find the values of the constants A, B and C. (4) 2 4x + 7x b Hence, or otherwise, expand ___________ in ascending powers of x, as far as the term in x2. (x − 2)(x + 4) Give each coefficient as a simplified fraction. (6) ⟶ ⟶ 10 OAB is a triangle. OA = a and OB = b. The points M and N are midpoints of OB and BA respectively. The triangle midsegment theorem states that 'In a triangle, the line joining the midpoints of any two sides will be parallel to the third side and half its length.' B M N O Use vectors to prove the triangle midsegment theorem. 362 A (4) Exam-style-practice 11 The diagram shows the region R bounded by the x-axis and the curve with equation 3π y = x2(sin x + cos x), 0 < x < ___ 4 y y = x2(sin x + cos x) 3π 4 O x The table shows corresponding values of x and y for y = x2(sin x + cos x). x 0 π __ π __ 3π ___ y 0 0.20149 0.87239 1.81340 8 4 8 π __ 5π ___ 3π ___ 2.08648 0 2 8 4 a Copy and complete the table giving the missing values for y to 5 decimal places. (1) b Using the trapezium rule, with all the values for y in the completed table, find an approximation for the area of R, giving your answer to 3 decimal places. (4) c Use integration to find the exact area of R, giving your answer to 3 decimal places. (6) d Calculate, to one decimal place, the percentage error in your approximation in part b. (1) 12 Ruth wants to save money for her newborn daughter to pay for university costs. In the first year she saves £1000. Each year she plans to save £150 more, so that she will save £1150 in the second year, £1300 in the third year, and so on. a Find the amount Ruth will save in the 18th year. (2) b Find the total amount that Ruth will have saved over the 18 years. (3) Ruth decides instead to increase the amount she saves by 10% each year. c Calculate the total

[illegible]

f(1.5245) = −0.0050… < 0 c x 0 = 2.5, x 1 = 2.6275, x 2 = 2.6406, x 3 = 2.6419, x 4 = 2.6420 −1.10670714 d 0.3 − ___ = 0.208 −12.02597883 a R = 0.37, α = 1.2405 2x b v9(x) = −0.148 sin(___ + 1.2405) 5 c v9(4.7) = −0.00312… < 0, v9(4.8) = 0.002798… > 0. Sign change implies maximum or minimum. d 12.607 e v9(12.60665) = 0.0000037… > 0, v9(12.60675) = −0.0000022… < 0. Sign change implies maximum or minimum. a=1 a cos 7x + cos 3x = cos (5x + 2x) + cos (5x – 2x) = cos 5x cos 2x – sin 5x sin 2x + cos 5x cos 2x + sin 5x sin 2x = 2 cos 5x cos 2x b __37 sin 7x + sin 3x + c m=3 24 16 __ 3√ 3 ____ 1− 26 __19 (e3 + 10) 8 5x + 3 3 1 a ___ ≡ ___ + ___ b ln 54 (2x − 3)(x − 2) 2x − 3 x + 2 Area = 2 a p = 1.350, q = 1.3818 (3 d.p.) b 2.5888 (4 d.p.) a __14 e2 + __14 2 − 2a a a (0, 0) and (____ , __) 5 5 dx a2 a b ___ = 0 ⇒ y = 2x + __ ⇒ 5 x 2 + 2ax + ___ = 0 2 4 dy b 2 − 4ac = 4a 2 − 5a 2 = −a 2 < 0 __ 3√ 3 421421 Answers Exam-style practice: Paper 1 1 2 3 4 5 6 7 8 dy ___ 2 sec t 1 ___ = ___ = cosec t sec 3 t = 2 dx 2 sin t cos t sin t cos 3 t a x > − __32 b x < −4, x > −1 c x > −1 a 2x + y – 3 = 0 → y = 3 – 2x x2 +kx + y2 + 4y = 4 x2 + kx + (3 – 2x)2 + 4(3 – 2x) = 4 5x2 + kx – 20x + 17 = 0 5x2 + (k___ – 20)x + 17 > 0 for ___ no intersections. b 20 − 2√ 85 < k < 20 + 2√ 85 Let f(x) = cos x f(x + h) − f(x) cos(x + h) − cos x f9(x) = lim ___ = lim ___ h→0 h→0 h h cos x cos h − sin x sin h − cos x ___ = lim h→0 h cos h − 1 sin h = lim ___ cos x − ___ sin x = − sin x) (h)] h→0 [(h a p = 4, p = −4 b Use p = −4, −18 432 49 ____ 705 ___ (− 8 , 64) a a u1 = a, u2 = 96 = ar, S∞ = 600 = ___ 1−r 96 ___ r So ___ = 600 ⇒ 96 = 600r(1 − r) ⇒ 96 = 600r 1−r − 600r 2 and therefore 25r2 – 25r + 4 = 0 b r = 0.2, 0.8 c a = 120 d n = 39 a y y = 2.79 − 0.01(x − 11) 2 , A = 2.79, B = 0.01, C = −11 11 m from goal, height of 2.79 m. 27.7 m (or 27.70 m) x = 0, y = 1.58. The ball will enter the goal. Surface area of box = 2x2 + 2(2xh + xh) = 2x2 + 6xh Surface area of lid = 2x2 + 2(6x + 3x) = 2x2 + 18x Total surface area = 4x2 + 6xh + 18x = 5356 5356 − 18x − 4 x 2 2678 − 9x − 2 x 2 So h = ___ = ___ 3x 6x V = 2x2h = __23 (2678x − 9 x 2 − 2 x 3) b 6x2 + 18x – 2678 = 0, x = 19.68 d 2V d 22 648.7 cm3 c ____2 < 0 ⇒ maximum e 31.7% dx 12 a b c d 13 a Exam-style practice: Paper 2 1 2 3 4 5 O −14 B9 (−6, 11) (0, 6) A C (−10, 0) (−3, 0) O b E (4, 0) 6 7 x y B (−6, 16) A (0, 5) D9 8 E (4, 5) C (−3, 5) O D (0, −1) 9 x 10 c y C (0, 0) O A (−7, 0) D (3, 12) 11 12 13 E (7, 0) x 14 B (−3, −22) x = 0.74, x = 5.54 dV 10 a ___ = −kV ⇒ ln V = −kt + c ⇒ V = V 0 e −kt dt 5 1 b k = __ ln(__), V0 = £35 100 c t = 11.45 years 3 3 11 a 14.9 miles b It is unlikely that a road could be built in a straight line, so the actual length of a road will be greater than 14.9 miles. 9 422422 a = −1, b = −4, c = 4 a y = −4x + 28 b y = __14 x + __52 c R(−10, 0) d 204 units2 1 __ ( ) y = 3 ln x + 1 , x > −1 a Student did not apply the laws of logarithms correctly in moving from the first line to the second line: 4A + B ≠ 4A + 4B b x = −2. Note x ≠ −5 y a −6 −6 10 x b −6 < y < 18 c a = −7, a = 0 __ __ a k=4 b x = −2, x = 3 + √ 7 , x = 3 − √ 7 a Area = __12 (x − 10)(x − 3) sin 30° = __14 (x 2 − 13x + 30) = 11 So x2 −13x + 30 = 44, and x2 – 13x – 14 = 0 b x = 14 (x ≠ 1, as x – 10 and x – 3 must be positive.) 13π a h = −5, k = 2, c = 36 b ____ 2 23 7 a A = 4, B = 5, C = −6 b − __ x − ___ x 2 8 32 ⟶ ⟶ OA = a and OB = b ⟶ ⟶ ⟶ 1⟶ 1⟶ 1 MN = MB + BN = __2 OB + __2 BA = __2 b + __12 (− b + a) = __12 a ⟶ ⟶ ⟶ ⟶ Therefore OA and MN are parallel and MN = __12 OA as required. a 2.46740 b 2.922 c 3.022 d 3.3% a £3550 b £40 950 c £51 159.09 a R = 0.41, α = 1.3495 b 40 cm at time t = 2.70 seconds c 0.38 seconds and 5.02 seconds. a h9(t) = 3e −0.3(x − 6.4) − 8e 0.8(x − 6.4) b __38 e −0.3(t − 6.4) = e 0.8(t − 6.4) ⇒ ln(__38 e −0.3(t − 6.4)) = 0.8(t − 6.4) ⇒ __45 ln(__38 e −0.3(t−6.4)) = t − 6.4 ⇒ t = __54 ln(__38 e −0.3(t−6.4)) + 6.4 c t0 = 5, t1 = 5.6990, t2 = 5.4369, t3 = 5.5351, t4 = 5.4983 d h9(5.5075) = 0.000360…> 0, h9(5.5085) = −0.000702… < 0. Sign change implies slope change, which implies a turning point. Full worked solutions are available in SolutionBank. Online Answers

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